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लक्ष्य विश्वमानम्

WEST BENGAL STATE UNIVERSITY

B.Sc. Honours PART-II Examinations, 2016

MATHEMATICS-HONOURS

PAPER-MTMA-III

Time Allotted: 4 Hours — Full Marks: 100

The figures in the margin indicate full marks. Candidates should answer in their own words and adhere to the word limit as practicable.

Group-A

Answer any *three* questions from the following: $5\times3=15$

1. Solve the equation by using Cardan's method: $x^3-30x+133=0$. 5

2. Solve the equation by Ferraris method: $2x^4+6x^3-3x^2+2=0$. 5

3. Find the special roots of the equation $x^{24}-1=0$ and deduce that $\cos\frac{\pi}{12}=\frac{\sqrt{3}+1}{2\sqrt{2}}$ and $\cos\frac{5\pi}{12}=\frac{\sqrt{3}-1}{2\sqrt{2}}$. 5

4. Determine the transformation $x=my+n$ which will change the equation $x^4+5x^3+9x^2+5x-1=0$ into reciprocal equation and hence solve it. 5

5. If $x_1, x_2, \ldots, x_n$ are positive real numbers not all equal and m is a rational number such that $0 < m < 1$ then prove that

$$\left(\frac{x_1^m + x_2^m + \cdots + x_n^m}{n}\right) < \left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right)^m.$$

6. (a) Find the least value of $x^2 + y^2 + z^2$ for positive values of x, y, z such that $2x + 3y + 6z = 14$.

(b) If a, b, c be all positive then show that

$$(a^2b + b^2c + c^2a)(ab + bc + ca) \geq abc(a + b + c)^2.$$

Group-B

Answer any *one* question from the following: 10×1

7. (a) If H and K be two subgroups of a group G then the product HK is a subgroup of G if $HK = KH$.

(b) Show that all roots of the equation $x^6 = 1$ form a cyclic group of order 6 under usual multiplication of complex numbers.

(c) What do you mean by an even permutation on the set $I_n = \{1, 2, \ldots, n\}$? Is the permutation $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 5 & 2 & 9 & 1 & 7 & 6 & 4 & 8 \end{pmatrix}$ an even permutation on I_9? Justify your answer.

8. (a) In the symmetric group S_3 show that the subsets $A = \{e, (1\ 2)\}$, $B = \{e, (1\ 2\ 3), (1\ 3\ 2)\}$ are subgroups. Use Lagrange's theorem to show that $A \cup B$ is not a subgroup of S_3.

(b) Prove that every subgroup of a cyclic group is cyclic.

(c) List all subgroup of a cyclic group of order 60.

(d) If G is a finite group then for any $a \in G$ prove that $a^{0(G)} = e$.

Group-C

Answer any *two* questions from the following: 10×2 = 20

9. (a) Define a vector space over a field F. 2

(b) Consider the subspaces 3+1

$S=\{(x,y,z,w)\in\mathbb{R}^4:\ 2x+y+z+w=0\}$

$T=\{(x,y,z,w)\in\mathbb{R}^4:\ x+2y+z+w=0\}$ of $\mathbb{R}^4$. Determine a basis of the subspace $S\cap T$ and hence determine the dimension of $S\cap T$.

(c) Prove that the matrix $\begin{pmatrix}2 & 4\\ 3 & 1\end{pmatrix}$ is diagonalisable and hence determine a diagonal matrix which is similar to it. 4

10.(a) State and prove Cayley-Hamilton theorem. 4

(b) Let $S=\{\alpha+\beta,\ \alpha-\beta,\gamma\}$, $T=\{\alpha,\ \beta+\gamma,\ \beta-\gamma\}$ be subsets of a real vector space V. Prove that $L(S)=L(T)$. 3

(c) Prove or disprove: Union of two subspaces of a vector space V is a subspace of V. 3

11.(a) Prove that (1, 1, 1) may replace any one of the vectors of the basis {(1, 2, 3), (2, 3, 1), (3, 1, 2)} of $\mathbb{R}^3$. 3

(b) Prove that each eigenvalue of a real orthogonal matrix has unit modulus. 3

(c) Find the row space and row rank of the matrix 4

$$A=\begin{pmatrix}1 & 1 & -2 & 1\\ 3 & 0 & 4 & 1\\ -1 & 2 & 5 & 2\end{pmatrix}.$$

12.(a) Prove that any singleton set consisting of non-zero vector of a finite dimensional vector space V over F is either a basis or can be extended to a basis of V. 4

(b) Prove that in an Euclidean vector space, $\|\alpha+\beta\| \le \|\alpha\|+\|\beta\|$. What happen when the equality holds? Give an example to show that if α, β are linearly dependent then $\|\alpha+\beta\|$ may not be equal to $\|\alpha\|+\|\beta\|$. 2+1+1

(c) Let $u=(x_1, x_2, x_3)$, $v=(y_1, y_2, y_3)$ be any two elements of $\mathbb{R}^3$. A mapping $f\colon \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ defined by $f(u,v)=x_1y_1+x_2y_2-x_3y_3$. Examine whether f is an inner product in $\mathbb{R}^3$. 2

Group-D

Answer any *two* questions from the following: $10\times 2=20$

13.(a) Prove that a monotone sequence cannot have two subsequences one of which is convergent and the other is divergent. 4

(b) Prove that the set of all subsequential limits of a bounded sequence is closed. 3

(c) Let $\{x_n\}$ and $\{y_n\}$ be two bounded real sequences such that $\{x_n\}$ converges. Prove that 3

$$\overline{\lim_{n\to\infty}}(x_n+y_n)=\lim_{n\to\infty}x_n+\overline{\lim_{n\to\infty}}y_n .$$

14.(a) Prove that the p-series $\sum\frac{1}{n^p}$ is convergent for $p>1$ and is divergent for $0<p\le 1$. 3

(b) State and prove the Leibnitz test for the convergence of an alternating series of reals. 1+3

(c) If $f(x+y)=f(x)+f(y)$ for all $x, y \in \mathbb{R}$ and $f(x)$ is continuous at a point of $\mathbb{R}$, prove that f is uniformly continuous on $\mathbb{R}$. 3

15.(a) State Riemann's re-arrangement theorem (on series). 2

(b) State Bolzano's theorem on continuous function. Does the result hold if the function is not continuous? 1+2

(c) If f and g be continuous on the interval $[a, b]$ such that $f(r) = g(r)$ for all rational values of r in $[a, b]$, then examine whether $f(x) = g(x)$ for all $x \in [a, b]$. 3

(d) Prove that if a function f is continuous and strictly monotonic increasing in $[a, b]$ then the function f is invertible. 2

16.(a) State and prove Darboux theorem on derivative. 5

(b) If α, β be the roots of the equation $ax^2 + bx + c = 0$, then find $\lim_{x \to \alpha} \frac{1 - \cos(ax^2 + bx + c)}{(x-\alpha)^2}$. 2

(c) If $y = a\cos(\log x) + b\sin(\log x)$, 3

prove that $x^2 y_{n+2} + (2n+1)xy_{n+1} + (n^2+1)y_n = 0$.

Group-E

Answer any *five* questions from the following: $5\times5 = 25$

17. $S = \{(x, y) \in \mathbb{R}^2 : x, y \in Q\}$. Show that S is neither open nor closed. 5

18. Prove that if $f(x, y)$ is continuous at (a, b) then $f(x, b)$ and $f(a, y)$ are continuous at $x = a$ and $y = b$ respectively. Is the converse true? 3+2

19. Define $f(x,y) = \begin{cases} x\sin\frac{1}{x} + y\sin\frac{1}{y}; & xy \neq 0 \\ 0 \quad ; & xy = 0 \end{cases}$ 2+1+1+1

Show that $\lim_{(x,y)\to(0,0)} f(x,y)$ exist but the repeated limits do not exist. Is $f(x, y)$ continuous at (0, 0)?

20. Let $f(x,y)=\begin{cases}(x^2+y^2)\log(x^2+y^2); & (x,y)\neq(0,0)\\ 0 & ;\ (x,y)=(0,0)\end{cases}$ 5

Show that f does not satisfy all the conditions of Schwarz theorem but $f_{xy}(0,0)=f_{yx}(0,0)$.

21. Transfer the equation $y\frac{\partial z}{\partial x}-x\frac{\partial z}{\partial y}=(y-x)z$ by introducing new independent variables $u=x^2+y^2, v=\frac{1}{x}-\frac{1}{y}$ and the new function $w=\log z-(x+y)$. 5

22. If $f(x, y, z)$ is a homogeneous function of degree n ($\neq 1$) having continuous second order partial derivatives then show that 5

$$\begin{vmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{vmatrix} = \frac{(n-1)^2}{z^2}\begin{vmatrix} f_{xx} & f_{xy} & f_x \\ f_{yx} & f_{yy} & f_y \\ f_x & fy & \frac{nf}{n-1} \end{vmatrix}.$$

23. If w is a differentiable function of u and v, where $u=x^2-y^2-2xy$ and $v=y$, then prove that $(x+y)\frac{\partial w}{\partial x}+(x-y)\frac{\partial w}{\partial y}=0$ is equivalent to $\frac{\partial w}{\partial v}=0$. 5

24. Show that the function $u=\frac{x}{y-z}$, $v=\frac{y}{z-x}$, $w=\frac{z}{x-y}$ are dependent and find the relation between them. 2+3

25. Justify the existence and uniqueness of the implicit function $y=y(x)$ for the functional equation $2xy-\log(xy)=2e-1$ near $(1, e)$. Also find $\frac{dy}{dx}$ at $(1, e)$. 4+1

Group-F

Answer any *two* questions from the following: $5\times2=10$

26. Show the area bounded by $y^2=ax^3$ and the double ordinate is $\frac{2}{5}$ of the area of the rectangle formed by this ordinate and the abscissa.

27. Find the centre of gravity of the planer region bounded by the parabola $\sqrt{\frac{x}{a}}+\sqrt{\frac{y}{b}}=1$ and the coordinate axes.

28. Show that the volume of the solid obtained by revolving the cardioid $r=a(1+\cos\theta)$ about the initial line is $\frac{8}{3}\pi a^3$.

29. Prove that the moment of inertia of a solid right circular cone of height h and semi vertical angle α about its axis is $\frac{3}{10}mh^2\tan^2\alpha$.