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WEST BENGAL STATE UNIVERSITY

B.Sc. Honours PART-I Examinations, 2016

MATHEMATICS-HONOURS

PAPER-MTMA-II

Time Allotted: 4 Hours | Full Marks: 100

The figures in the margin indicate full marks. Candidates should answer in their own words and adhere to the word limit as practicable. All symbols are of usual significance.

Group-A

Answer any *five* questions from the following: $5\times5=25$

1. (a) Using Archimedean property of $\mathbb{R}$ prove that for any $y\in\mathbb{R}$, $y>0$, $\exists\ m\in\mathbb{N}$ such that $\frac{1}{2^m}<y$. 2

   (b) Let A and B be two non-empty bounded sets of real numbers : $a=\sup A$, $b=\sup B$. Let $C=\{x+y:x\in A, y\in B\}$, show that $C=a+b$. 2

   (c) Show that the well ordering property of natural numbers implies the principle of mathematical induction. 1

2. State and prove Cantor's theorem on nested intervals. 5

3. (a) Show that a monotonic increasing sequence which is bounded above is convergent. 2

(b) Find: $\lim \dfrac{(n+1)^{2n}}{(n^2+1)^n}$ 2

(c) State Sandwich theorem for sequences. 1

4. (a) Prove that the sequence $\{u_n\}$ satisfying the condition $|u_{n+2}-u_{n+1}| \le c|u_{n+1}-u_n|$ for all $n \in \mathbb{N}$, where $0 < c < 1$, is a Cauchy sequence. 2

(b) A sequence $\{x_n\}$ is defined as follows: $0 < x_1 < 1$ and $(2-x_n)x_{n+1}=1$, $\forall\, n \ge 1$. Show that $\{x_n\}$ converges to 1. 2

(c) Prove that every Cauchy sequence in $\mathbb{R}$ is bounded. 1

5. (a) Prove that every bounded infinite subset of $\mathbb{R}$ has at least one limit point in $\mathbb{R}$. 3

(b) Give an example of perfect set. Give an example of a denumerable collection of open sets whose intersection is again an open set. 1+1

6. (a) Show that the interior of a set is an open set. 2

(b) Prove that no non-empty proper subset of $\mathbb{R}$ is both open and closed in $\mathbb{R}$. 3

7. (a) Construct an infinite subset of $\mathbb{R}$ having exactly two isolated points. 1

(b) Let $S \subseteq \mathbb{R}$ and f, g are two real valued functions on S, $c \in S'$. If f is bounded on $N'(c) \cap S$ for some deleted *nbd.* $N'(c)$ of c and $\lim_{x\to c} g(x)=0$, then prove that $\lim_{x\to c}(f\cdot g)(x)=0$. 3

(c) State Cauchy's criterion for the existence of finite limit of a function. 1

8. (a) Let $D \subset R$ and $f: D \to R$ be a function. If c is an isolated point of D, then show that f is continuous at c. 2

(b) Show that $\lim_{x\to\infty} \frac{[x]}{x} = 1$, where $[x]$ has its usual meaning. 2

(c) Define uniform continuity. 1

9. (a) Prove that the Dirichlet's function f defined on $\mathbb{R}$ by 2

$$f(x)=\begin{cases} -1 & \text{when } x \text{ is rational} \\ 1 & \text{when } x \text{ is irrational} \end{cases}$$

is discontinuous at every point.

(b) If a function $f: [a, b] \to \mathbb{R}$ is monotone on $[a,b]$ then prove that the set of points of discontinuities of f in $[a,b]$ is a countable set. 3

Group-B

10. Answer any *two* questions from the following: $4\times2=8$

(a) If $I_{m,n} = \int_0^{\pi/2} \cos^m x \sin nx\, dx$, then show that $I_{m,n} = \frac{1}{2^{m+1}}\left[2+\frac{2^2}{2}+\frac{2^3}{3}+\cdots+\frac{2^m}{m}\right]$. 4

(b) If $u_n = \int_0^{\pi/2} \theta \sin^n \theta\, d\theta$ and $n>1$, then prove that $u_n = \frac{n-1}{n}u_{n-2}+\frac{1}{n^2}$. 4

(c) For $m>-1$, $n>-1$, prove that 4

$$\int_a^b (x-a)^m (b-x)^n\, dx = (b-a)^{m+n+1}\frac{\Gamma(m+1)\cdot\Gamma(n+1)}{\Gamma(m+n+2)}.$$

11. Answer any *three* questions from the following: $4\times3 = 12$

(a) Determine the pedal equation of $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with respect to a focus where $a^2 > b^2$. 4

(b) Find the evolute of the curve $x=a(1+\cos^2 t)\sin t$, $y=a\sin^2 t\cos t$. 4

(c) Find the asymptote, if any $y=\frac{x^2+1}{\sqrt{x^2-1}}$. 4

(d) Show that the points of inflection of the curve $y(x^2+a^2)=a^2x$ lie on a straight line. 4

(e) Find the envelope of the family of ellipses $\frac{(x-h)^2}{\alpha^2}+\frac{(y-k)^2}{\beta^2}=1$, where the parameters h, k are connected by the relation $\frac{h^2}{\alpha^2}+\frac{k^2}{\beta^2}=1$. 4

Group-C

Answer any *three* questions from the following $10\times3 = 30$

12.(a) Examine whether the equation $xydx + (2x^2 + 3y^2 - 20)dy$ is exact or not. Hence solve it. 1+2

(b) Reduce the equation 1+3

$$\sin y\frac{dy}{dx}=\cos x(2\cos y-\sin^2 x)$$ to a linear equation and hence solve it.

(c) Find the orthogonal trajectories of $\frac{a}{r}=1+\cos\theta$, a being a parameter. 3

13.(a) Transform the differential equation $x^2p^2 + py(2x + y) + y^2 = 0$ to Clairaut's form by substituting $y = u$, $xy = v$. Hence find the general and singular solution. 1+2+2

(b) Solve : $\sec^2 y \frac{dy}{dx} + 2x \tan y = x^3$. 5

14.(a) Solve by the method of undetermined coefficient : 5

$(D^2 - 2D + 3)y = x^3 + \sin x$.

(b) Solve : $(D^2 + 4)y = x \sin^2 x$. 5

15.(a) Solve by the method of variation of parameters : 5

$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = \frac{e^{3x}}{x^2}$.

(b) Solve: $x^3 \frac{d^3y}{dx^3} + 2x^2 \frac{d^2y}{dx^2} + 2 = 10\left(x + \frac{1}{x}\right)$. 5

16.(a) Solve $\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + (4x^2 - 3)y = e^{x^2}$ by reducing to normal form. 5

(b) Show that $(1 + x + x^2)\frac{d^3y}{dx^3} + (3 + 6x)\frac{d^2y}{dx^2} + 6\frac{dy}{dx} = 0$ is exact and solve it. 5

17.(a) Solve: $(x+2)\frac{d^2y}{dx^2} - (2x+5)\frac{dy}{dx} + 2y = (1+x)e^x$ by the method of operational factors. 5

(b) Solve $\frac{d^2y}{dx^2} - \frac{1}{x}\frac{dy}{dx} + 4x^2y = x^4$ by changing the independent variable. 5

Group-D

Answer any *five* questions from the following $5\times5=25$

18. ABC is a triangle and D, E, F are points on the sides BC, CA and AB respectively such that BD = $\frac{1}{3}$BC, CE = $\frac{1}{3}$CA, AF = $\frac{1}{3}$ AB. Show that area of the triangle ABC is equal to three times the area of the triangle DEF. 5

19. If the internal and external bisectors of the angle $\angle A$ of triangle ABC meet the opposite side BC at D and E respectively, show that BD, BC and BE are in harmonic progression. 5

20.(a) If $\vec{a},\vec{b},\vec{c}$ be any three non-coplanar vectors then show that any vector $\vec{r}$ can be expressed as $\vec{r}=\frac{[\vec{r}\,\vec{b}\,\vec{c}]}{[\vec{a}\,\vec{b}\,\vec{c}]}\vec{a}+\frac{[\vec{a}\,\vec{r}\,\vec{c}]}{[\vec{a}\,\vec{b}\,\vec{c}]}\vec{b}+\frac{[\vec{a}\,\vec{b}\,\vec{r}]}{[\vec{a}\,\vec{b}\,\vec{c}]}\vec{c}$ 3

(b) Show by vector method that the points (2, 1, 4), (3, –1, 7), (0, 4, 0) and (2, 0, 6) are coplanar. 2

21. If $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{a}'$, $\vec{b}'$, $\vec{c}'$ are reciprocal system of vectors then prove that 3+2

(i) $\vec{a}'\times\vec{b}'+\vec{b}'\times\vec{c}'+\vec{c}'\times\vec{a}'=\frac{1}{[\vec{a}\,\vec{b}\,\vec{c}]}(\vec{a}+\vec{b}+\vec{c})$.

(ii) $\vec{a}\cdot\vec{a}'+\vec{b}\cdot\vec{b}'+\vec{c}\cdot\vec{c}'=3$.

22. Find the vector equation of the plane passing through the point $5\vec{i}+2\vec{j}-3\vec{k}$ and perpendicular to each of the planes $\vec{r}\cdot(2\vec{i}-\vec{j}+2\vec{k})=2$ and $\vec{r}\cdot(\vec{i}+3\vec{j}-5\vec{k})=5$. 5

23. A (1, 0, 1), B (1, 1, 0), C (2, –1, 1) are three points. Find, by vector method, the locus of a point P if the volume of the tetrahedron PABC is 2 units. 5

24.(a) Prove, by vector method, that three concurrent forces represented in magnitude and directions by the medians of a triangle are in equilibrium. 3

(b) A particle acted on by constant forces $4\vec{i}+5\vec{j}-3\vec{k}$ and $3\vec{i}+2\vec{j}+4\vec{k}$ is displaced from the point $\vec{i}+3\vec{j}+\vec{k}$ to the point $2\vec{i}-\vec{j}+3\vec{k}$. Find the total work done by the forces. 2

25. Prove that curl curl $\vec{F}=\vec{\nabla}(\vec{\nabla}\cdot\vec{F})-\nabla^2\vec{F}$. 5

26. Show that the vector $\vec{F}=(2x-yz)\vec{i}+(2y-zx)\vec{j}+(2z-xy)\vec{k}$ is irrotational. Also find a scalar function φ such that $\vec{F}=\text{grad}\varphi$. 2+3