MAT (H) – II
P – III

MTMA(HN)-03

West Bengal State University

B.A./B.Sc./B.Com (Honours, Major, General) Examinations, 2015

PART – II
MATHEMATICS — HONOURS
Paper – III

Duration : 4 Hours] [Full Marks : 100

The figures in the margin indicate full marks.

Group – A

Answer any *three* questions. $3 \times 5 = 15$

1. Solve the equation $3x^6 + x^5 - 27x^4 + 27x^2 - x - 3 = 0$.

2. Solve the equation by Cardan's method :

$$28x^3 - 9x^2 + 1 = 0$$

3. Prove that special roots of the equation $x^9 - 1 = 0$ are the roots of the equation $x^6 + x^3 + 1 = 0$ and their values are $\cos \frac{2r\pi}{9} + i \sin \frac{2r\pi}{9}$, $r = 1, 2, 4$.

4. Solve by Ferraris' method :

$$x^4 + 2x^3 - 7x^2 - 8x + 12 = 0$$

5. a) If a, b are positive rational numbers and $a > b$ then prove that $a^{2a} < (a+b)^{a+b}(a-b)^{a-b}$. 3

b) State Cauchy-Schwarz inequality. 2

6. a) If a, b, c are all positive and $abc = k^3$ then prove that $(1+a)(1+b)(1+c) \geq (1+k)^3$. 2

b) If $a_1, a_2, \ldots, a_n$ are all positive and $S = a_1 + a_2 + \ldots + a_n$ then prove that

$$\frac{S}{S-a_1} + \frac{S}{S-a_2} + \ldots + \frac{S}{S-a_n} \geq \frac{n^2}{n-1}.$$ 3

Group – B

Answer any *one* question. $1 \times 10 = 10$

7. a) Show that A_3 the set of all even permutations of { 1, 2, 3 } is a cyclic group with respect to product of permutations. Is it commutative ? Answer with reason. 4 + 1

b) If (G, o) be an infinite cyclic group generated by a then prove that a and a^{-1} are the only generators of the group. 3

c) Write down the permutation $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 7 & 6 & 2 & 1 & 3 & 4 \end{pmatrix}$ as a product of disjoint cycles and then express it as a product of transpositions. 1 + 1

8. a) Let H be a subgroup of a group G. Then prove that the set of all left cosets of H in G and the set of all right cosets of H in G have the same cardinality. 4

b) Prove that every group of prime order is cyclic. 3

c) Show that if two right cosets Ha and Hb be distinct then two left cosets $a^{-1}H$ and $b^{-1}H$ are distinct. 2

d) Find the order of the permutation $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 1 & 3 & 2 \end{pmatrix}$. 1

Group – C

Answer any *two* questions. $2 \times 10 = 20$

9. a) Define basis of a vector space. Prove that if $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be a basis of a finite dimensional vector space V then any set of linearly independent vectors of V contains atmost n vectors. 2 + 2

b) Let V be a vector space of all real matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and let W be a subset of those matrices for which $a + b = 0$. Prove that W is a subspace of V. Find a basis of W. 2 + 2

c) Find the coordinates of the polynomial $(x - 3x^2)$ relative to the ordered basis $\{1 - x, 1 + x, 1 - x^2\}$ in the vector space P_2 of all polynomials of degree at most 2 over the field of real numbers. 2

10. a) Let A & B be two matrices over the same field F such that AB is defined. Then prove that rank $(AB) \le \min\{\operatorname{rank}A, \operatorname{rank}B\}$. 5

b) If row rank of the matrix

$$A = \begin{bmatrix} 3 & 4 & -3 & 5 \\ 1 & 2 & -1 & 7 \\ 4 & 1 & 2 & 9 \\ 2 & -1 & 4 & k \end{bmatrix}$$

be three then find the value of k. 5

11. a) If α and β be any two vectors in an inner product space $V(F)$ then prove that $|(\alpha, \beta)| \le \|\alpha\| \|\beta\|$. 4

b) If α, β be vectors in a real inner product space and $\|\alpha\| = \|\beta\|$, then show that $(\alpha + \beta, \alpha - \beta) = 0$. 3

c) Prove that eigenvectors corresponding to two distinct eigenvalues of a real symmetric matrix are orthogonal. 3

12. a) Show that the matrix $A = \begin{pmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{pmatrix}$ is diagonalisable. 5

b) Apply Gram-Schmidt orthogonalisation process to the set of vectors $\{(1,-1, 1), (2, 0, 1), (0, 1, 1)\}$ to obtain an orthogonal basis of $\mathbb{R}^3$ with standard inner product. 5

Group – D

Answer any *two* questions. $2 \times 10 = 20$

13. a) If a sequence $\{x_n\}$ converges to l, then prove that every subsequence of $\{x_n\}$ also converges to l. 3

b) State and prove Bolzano-Weierstrass theorem on subsequence. 4

c) Show that the sequence $\{a_n\}$ defined by $a_n = \left(1 - \frac{1}{n}\right) \sin \frac{n\pi}{2}$, $n = 1, 2, \ldots$ has convergent subsequence but the sequence is not convergent. 3

14. a) Use Cauchy's condensation test to show that $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$ converges for $p > 1$ and diverges for $0 < p \leq 1$. 3

b) Test the convergence of the series

$$\frac{a}{b} + \frac{1+a}{1+b} + \frac{(1+a)(2+a)}{(1+b)(2+b)} + \ldots.$$ 3

c) State and prove Leibnitz test for an alternating series. 4

15. a) If f is continuous on $[a, b]$ and $f(a) \neq f(b)$ then prove that f takes every value between $f(a)$ and $f(b)$ at least once in (a, b). 4

b) Prove that a real valued continuous function on a closed bounded interval I is bounded on I. 2

c) If $f:[a,b] \to R$ be such that f has a local maximum at an interior point c of $[a, b]$ and $f'(c)$ exists, then prove that $f'(c) = 0$. 2

d) Find $\lim\limits_{x \to 1-} (1-x)^{\cos\frac{\pi x}{2}}$. 2

16. a) State and prove Taylor's theorem with Lagrange's form of remainder. 4

b) Prove that $\frac{2x}{\pi} < \sin x < x$ when $0 < x < \frac{\pi}{2}$. 2

c) If f'' is continuous on some *nbd* of c, then prove that $\lim\limits_{h \to 0} \frac{f(c+h) - 2f(c) + f(c-h)}{h^2} = f''(c)$. 2

d) Show that the function $f(x) = x^{1/x}$ has a maximum at $x = e$. 2

Group – E

Answer any *five* questions. 5 × 5 = 25

17. Show that

$$f(x, y) = \begin{cases} \frac{x^3 - y^3}{x^2 + y^2}, (x, y) \neq (0,0) \\ 0, \quad \text{when } x^2 + y^2 = 0 \end{cases}$$

is continuous at $(0, 0)$, but it is not differentiable at $(0, 0)$.

18. Let (a, b) be an interior point of domain of definition of a function f of two variables x, y. If $f_x(a,b)$ exists and $f_y(x,y)$ is continuous at (a, b), then prove that $f(x, y)$ is differentiable at (a, b).

19. If

$$f(x,y) = \begin{cases} xy, & \text{when } |x| \geq |y| \\ -xy, & \text{when } |x| < |y| \end{cases}$$

Show that $f_{xy}(0,0) \neq f_{yx}(0,0)$.

Which condition of Schwarz's theorem is not satisfied by f ?

20. State and prove the converse of Euler's theorem on homogeneous function of three variables.

21. If a function $f(x, y)$ of two variables x and y when expressed in terms of new variables u and v defined by $x = \frac{1}{2}(u+v)$ and $y^2 = uv$ becomes $g(u, v)$, then show that $\frac{\partial^2 g}{\partial u \partial v} = \frac{1}{4}\left(\frac{\partial^2 f}{\partial x^2} + \frac{2x}{y}\frac{\partial^2 f}{\partial x \partial y} + \frac{\partial^2 f}{\partial y^2} + \frac{1}{y}\frac{\partial f}{\partial y}\right)$.

22. Given $f(x+y) = \frac{f(x)+f(y)}{1-f(x).f(y)}$; $f(x).f(y) \neq 1$, where x and y are independent variables and $f(t)$ is a differentiable function of t and $f(0) = 0$. Using the property of Jacobian, show that $f(t) = \tan \alpha t$, where α is a constant.

23. Using the method of Jacobian, show that the functions $u = x + y - z$, $v = x - y + z$, $w = x^2 + y^2 + z^2 - 2yz$ are dependent. Find also the relation between them.