West Bengal State University

B.A./B.Sc./B.Com (Honours, Major, General) Examinations, 2015

PART - I

MATHEMATICS — HONOURS

Paper - II

Duration : 4 Hours] [Full Marks : 100

The figures in the margin indicate full marks.

GROUP - A

(Marks : 25)

Answer any *five* questions. 5 × 5 = 25

1. a) State well ordering property of natural numbers and the principle of mathematical induction. Verify whether the well-ordering property is true on Z, the set of integers ? 1 + 2

   b) Prove that the set $\mathbb{N}$ is not bounded above. 2

2. a) State Cauchy's general principle of convergence and use it to prove that the sequence $\{x_n\}$ is convergent, where $x_n = 1 + \frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{n!}$. 1 + 3

   b) State Cauchy's first theorem on limits. 1

3. a) Is the density property of an ordered field implies the order of completeness ? Justify your answer. 1 + 2

   b) Find $Sup\, A$, where $A = \{ x \in \mathbb{R} : 3x^2 + 8x - 3 < 0 \}$. 1

   c) Prove that l is an interior point of $S \in R$ implies l is a limit point of S. 1

4. a) State Cauchy's second limit theorem on sequence. Is the converse of the Cauchy's second limit theorem true ? Justify your answer. 1 + 1

b) Using Cauchy's first limit theorem prove that $\left\{ \frac{1 + \sqrt[2]{2} + \sqrt[3]{3} + \ldots + \sqrt[n]{n}}{n} \right\}$ converges to 1. 2

c) Give an example of two non-convergent sequences $\{x_n\}$ and $\{y_n\}$ such that $\{x_n + y_n\}$ is convergent. 1

5. a) Prove that every infinite subset of a denumerable set is denumerable. 3

b) Give examples one each of a denumerable set and a non-denumerable set. 2

6. State and prove Bolzano-Weierstrass theorem on accumulation points. 5

7. a) Let $f : S \to \mathbb{R}$, $S \in R$ be a function, c be a limit point of S. Let $\lim_{x \to c} f(x) = l$. Prove that for every sequence $\{x_n\}$ in $S - \{c\}$ converging to c, the sequence $\{f(x_n)\}$ converges to l. 2

b) Prove that $\lim_{x \to 0+} \sqrt{x} \sin \frac{1}{x} = 0$. 2

c) Evaluate $\lim_{x \to 3} [x] - \left[\frac{x}{3} \right]$, where $[x]$ is the greatest integer not exceeding x. 1

8. a) Let $f : S \to \mathbb{R}$, $S \in \mathbb{R}$ be continuous on S, $c \in S$ and $f(c) < 0$. Then prove that there exists a neighbourhood of c, $N_\delta(c)$, $\delta > 0$, such that

$f(x) . f(c) > 0, \quad \forall x \in N_\delta(c)$. 2

b) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and U be an open set in $\mathbb{R}$. Prove that $f^{-1}(U)$ is also an open set in $\mathbb{R}$. 2

c) Give an example of a function $f : [0, 1] \to \mathbb{R}$ such that f is not continuous but $|f|$ is continuous. 1

9. a) Prove that the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 2x , & \text{when } x \in Q \\ 1 - x , & \text{when } x \in \mathbb{R} - Q \end{cases}$$

is continuous at $x = \frac{1}{3}$ and discontinuous at all other points. 3

b) Give an example of a function $f : [0, 1] \to \mathbb{R}$ such that f is continuous but not monotone on $[0, 1]$. 1

c) Give an example of discontinuity of second kind. 1

GROUP - B

(Marks : 20)

10. Answer any *two* of the following questions : $2 \times 4 = 8$

a) If $I_{m,n} = \int_0^1 x^m (1-x)^n \, dx \; (m, n \in \mathbb{N})$, prove that

$(m + n + 1) I_{m,n} = n I_{m,n-1}$ and hence find the value of $I_{m,n}$.

b) Prove that

$$I_{m,n} = \int \sin^m x \cos^n x \, dx = \frac{\sin^{m+1} x \cos^{n-1} x}{m+n} + \frac{n-1}{m+n} I_{m,n-2}.$$

c) Show that $2^{2m-1} \Gamma(m) \Gamma(m + \frac{1}{2}) = \sqrt{\pi} \, \Gamma(2m)$ $m > 0$.

11. Answer any *three* of the following questions : $3 \times 4 = 12$

a) Find the pedal equation of the cardioid $r = a(1 + \cos\theta)$. 4

b) Determine the rectilinear asymptotes, if any, of the curve $y = x + \log x$. 4

c) If ρ_1 and ρ_2 be the radii of curvature at the ends of a focal chord of the parabola $y^2 = 4ax$, then show that $\rho_1^{-2/3} + \rho_2^{-2/3} = (2a)^{-2/3}$. 4

d) Find the envelopes of the family of circles $x^2 + y^2 - 2ax - 2by + b^2 = 0$, where a, b are parameters, whose centres lie on the parabola $y^2 = 4ax$. 4

e) Find if there is any point of inflexion on the curve $y - 3 = 6(x-2)^5$. 4

GROUP - C

(Marks : 30)

Answer any *three* of the following questions. $3 \times 10 = 30$

12\. a) Define orthogonal trajectory. Find the orthogonal trajectories of the family of curves $y^2 = 4ax$, a being parameter $a > 0$. 1 + 4

b) Solve $\frac{dy}{dx} + \frac{x}{1-x^2} y = x\sqrt{y}$. 3

c) Find an integrating factor of the differential equation

$$(2xy^4e^y + 2xy^3 + y) dx + (x^2y^4e^y - x^2y^2 - 3x) dy = 0 .$$ 2

13\. a) Transform the given equation to Clairaut's equation by putting $x^2 = u$ and $y^2 = v$ and hence find the general and singular solutions :

$$(px - y) (x - py) = 2p, \text{ where } p = \frac{dy}{dx}.$$ 1 + 2 + 2

b) Solve : $\frac{dy}{dx} + \frac{y}{x} \log y = \frac{y}{x^2} (\log y)^2$. 5

14\. a) Solve : $\frac{d^2y}{dx^2} - y = e^x \sin \frac{x}{2}$. 5

b) Find the orthogonal trajectories of the family of coaxial circles

$x^2 + y^2 + 2gx + c = 0$, where g is a parameter and c is constant. 5

15\. a) Solve : $x^4 \frac{d^3y}{dx^3} + 3x^3 \frac{d^2y}{dx^2} - 2x^2 \frac{dy}{dx} + 2xy = \log x$. 5

b) Solve by the method of undetermined coefficients the differential equation

$(D^2 - 3D + 2) y = 14 \sin 2x - 18 \cos 2x$. 5

16. a) Solve $\sin^2 x \dfrac{d^2y}{dx^2} = 2y$, given that $\cot x$ is one of the solutions. 5

b) Solve $\dfrac{d^2y}{dx^2} - 4x\dfrac{dy}{dx} + (4x^2 - 1)y = -3e^{x^2} \sin 2x$ by reducing it to normal form. 5

17. a) Solve, by the method of variation of parameters

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = x^2 e^x.$$ 5

b) Solve $x\dfrac{d^2y}{dx^2} + (x-2)\dfrac{dy}{dx} - 2y = x^3$, by the method of operational factors. 5

GROUP - D

(Marks : 25)

Answer any *five* of the following questions. $5 \times 5 = 25$

18. Show, by vector method, that the straight line joining the mid-points of two non-parallel sides of a trapezium are parallel to the parallel sides and half of their sum in length. 5

19. Prove that the necessary and sufficient condition for three distinct points with position vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ to be collinear is that there exist three scalars x, y, z not all zero such that $x\vec{a} + y\vec{b} + z\vec{c} = \vec{0}$ and $x + y + z = 0$. 5

20. a) If $\vec{\alpha}$, $\vec{\beta}$, $\vec{\gamma}$ are three vectors such that $\vec{\alpha} + \vec{\beta} + \vec{\gamma} = \vec{0}$ and $|\vec{\alpha}| = 3$, $|\vec{\beta}| = 5$, $|\vec{\gamma}| = 7$, then find the angle between $\vec{\alpha}$ and $\vec{\beta}$. 3

b) Find the unit vector which is perpendicular to the vectors $3\vec{i} - 2\vec{j} - \vec{k}$ and $2\vec{i} - \vec{j} - 3\vec{k}$. 2

21. a) If $\vec{\alpha} \times \vec{\beta} + \vec{\beta} \times \vec{\gamma} + \vec{\gamma} \times \vec{\alpha} = \vec{0}$ then show that $\vec{\alpha}, \vec{\beta}, \vec{\gamma}$ are coplanar. 3

b) Find the vector equation of the plane passing through the origin and parallel to the vectors $2\vec{i} + 3\vec{j} + 4\vec{k}$ and $4\vec{i} - 5\vec{j} + 4\vec{k}$. 2

22. a) A particle acted on by two constant forces $\vec{i} + \vec{j} - 3\vec{k}$ and $3\vec{i} + \vec{j} - \vec{k}$ is displaced from the point $\vec{i} + 2\vec{j} + 3\vec{k}$ to the point $3\vec{i} + 4\vec{j} + 2\vec{k}$. Find the total work done. 3

b) Find the moment of the force $4\vec{i} + 2\vec{j} + \vec{k}$ acting at a point $5\vec{i} + 2\vec{j} + 4\vec{k}$ about the point $3\vec{i} - \vec{j} + 3\vec{k}$. 2

23. Show that $[\, \vec{\beta} \times \vec{\gamma} \quad \vec{\gamma} \times \vec{\alpha} \quad \vec{\alpha} \times \vec{\beta} \,] = [\, \vec{\alpha} \quad \vec{\beta} \quad \vec{\gamma} \,]^2$. 5

24. a) Find a simplified from of $\vec{\nabla} \times (\vec{r} f(r))$ where $f(r)$ is differentiable and $r = |\vec{r}|$. 2

b) Show that the vector $\dfrac{\vec{r}}{r^3}$, where $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ is both irrotational and solenoidal. 3

25. a) Find the directional derivative of the function $f(x, y, z) = yz + zx + xy$ in the direction of the vector $\vec{u} = \vec{i} + 2\vec{j} + 2\vec{k}$ at the point $(1, 2, 0)$. 3

b) Prove that $div\,(grad\, f) = \nabla^2 f$. 2

26. a) If $\vec{r} = a\vec{i}\cos t + a\vec{j}\sin t + bt\vec{k}$ then show that $[\dot{\vec{r}}\ \ddot{\vec{r}}\ \dddot{\vec{r}}] = a^2 b$. 3

b) If $\vec{w}$ is a constant vector, $\vec{r}$ and $\vec{s}$ are functions of a scalar variable t and if $\frac{d\vec{r}}{dt} = \vec{w} \times \vec{r}$ and $\frac{d\vec{s}}{dt} = \vec{w} \times \vec{s}$ then show that

$\frac{d}{dt}(\vec{r} \times \vec{s}) = \vec{w} \times (\vec{r} \times \vec{s})$. 2