West Bengal State University

B.A./B.Sc./B.Com. (Honours, Major, General) Examinations, 2014

PART-II

MATHEMATICS– Honours

Paper– IV

Duration : 4 Hours Full Marks : 100

Candidates are required to give their answers in their own words as far as practicable.

The figures in the margin indicate full marks.

Group-A

Answer any *two* questions. $2 \times 10 = 20$

1. a) Prove that the locus of the pole with respect to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ of any tangent to the auxiliary circle is the curve $\frac{x^2}{a^4}+\frac{y^2}{b^4}=\frac{1}{a^2}$. 5

   b) From points on the circle $x^2+y^2=a^2$ tangents are drawn to the hyperbola $x^2-y^2=a^2$. Prove that the locus of the middle points of the chords of contact is the curve $\left(x^2-y^2\right)^2=a^2\left(x^2+y^2\right)$. 5

2. a) Show that the condition that the plane $ax+by+cz=0$ may cut the cone $yz+zx+xy=0$ in the perpendicular lines is $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=0$. 5

   b) Find the locus of the point of intersection of the perpendicular generators of the hyperbolic paraboloid $\frac{x^2}{a^2}-\frac{y^2}{b^2}=2z$. 5

3. a) Reduce the equation $x^2+y^2+z^2-2xy-2yz+2zx+x-4y+z+1=0$ to its canonical form and determine the type of the quadric that it represents. 5

   b) Prove that the enveloping cylinder of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ whose generators are parallel to the straight line $\frac{x}{0}=\frac{y}{\pm\sqrt{a^2-b^2}}=\frac{z}{c}$ meets the plane z = 0 in circles. 5

Group - B

Answer any *one* question. $1 \times 10 = 10$

4. a) Find the eigenvalues and eigenfunctions of the differential equation $\frac{d}{dx}\left(x\frac{dy}{dx}\right)+\frac{\lambda}{x}y=0\ (\lambda>0)$ satisfying the boundary conditions

$y(1)=0$ and $y'(e)=0$. 5

b) Solve :

$$\frac{dx}{z(x+y)}=\frac{dy}{z(x-y)}=\frac{dz}{x^2+y^2}.$$ 5

5. a) Solve by Lagrange's method :

$$x^2(y-z)p+y^2(z-x)q=z^2(x-y),\ \left[p=\frac{\partial z}{\partial x},\ q=\frac{\partial z}{\partial y}\right].$$ 5

b) Solve by Charpit's method :

$$2x\left(q^2z^2+1\right)=pz.$$ 5

Group - C

Answer either Q. No.6 or Q.No. 7 and either Q.No. 8 or Q. No. 9.

$13 + 12 = 25$

6. a) Prove that the set of all feasible solutions of an L.P.P. is a convex set. 6

b) Solve the L.P.P.

Maximize $Z = 2x_1+3x_2+x_3$

subject to $-3x_1+2x_2+3x_3=8$

$-3x_1+4x_2+2x_3=7$

$x_1, x_2, x_3 \geq 0.$ 7

7. a) Use duality to solve the following L.P.P.

Minimize $Z=x_1-x_2$

subject to $2x_1+x_2 \geq 2$

$-x_1-x_2 \geq 1.$

$x_1, x_2 \geq 0$ 7

b) Prove that if either the primal or the dual problem has a finite optimal solution, then the other problem will also have a finite optimal solution and the optimal values of the objective functions in both the problems will be same. 6

8. a) Obtain an optimal basic feasible solution to the following transportation problem : 6

	w_1	w_2	w_3	w_4	a_i
F_1	19	30	50	10	7
F_2	70	30	40	60	9
F_3	40	8	70	20	18
b_j	5	8	7	14	

b) Solve the following travelling Salesman problem : 6

	A	B	C	D
A	∞	12	10	15
B	16	∞	11	13
C	17	18	∞	20
D	13	11	18	∞

9. a) In a rectangular game, the pay-off matrix A is given by

$$A = \begin{pmatrix} 3 & 2 & -1 \\ 4 & 0 & 5 \\ -1 & 3 & -2 \end{pmatrix}$$

State, with justifications, whether the players will use pure or mixed strategies. What is the value of the game ? 6

b) Solve the following game graphically or otherwise : 6

B

A	B_1	B_2
A_1	2	7
A_2	3	5
A_3	11	2

Group-D

Answer any *three* questions. $3 \times 15 = 45$

10. a) A particle is projected vertically upwards with a velocity u in a medium whose resistance varies as the square of the velocity. Investigate the motion. 7

b) One end of an elastic string whose modulus of elasticity is λ and whose unstretched length a is fixed to a point on a smooth horizontal table and the other end is tied to a particle of mass m which is lying on the table. The particle is pulled to a distance where the extension of the string is b and then let go. Show that the time of small oscillation is

$$2\left(\pi + \frac{2a}{b}\right)\sqrt{\frac{am}{\lambda}}.$$ 8

11. a) Find the tangential and normal components of velocities and accelerations of a particle which describes a plane curve. 7

b) One end of an elastic string of unstretched length a is tied to a point on the top of a smooth table and a particle attached to the other end can move freely on the table. If the path be nearly circular of radius b, show that its apsidal angle is approximately

$$\pi\sqrt{\left(\frac{b-a}{4b-3a}\right)}.$$ 8

12. a) Two smooth spheres of masses m_1 and m_2 moving with respective velocities u_1 and u_2 in the same direction impinge directly. If e be the co-efficient of restitution between them, find their velocities after impact, loss of kinetic energy and impulsive action. 7

b) A body describing an ellipse of eccentricity e under the action of a force tending to a focus, and when at the nearer apse, the centre of force is transferred to the other focus. Prove that the eccentricity of the new orbit is $\frac{e(3+e)}{1-e}$. 8

13. a) A particle is projected vertically upwards from the earth's surface with a velocity just sufficient to carry it to infinity. Prove that the time it takes in reaching a height h is

$$\frac{1}{3}\sqrt{\frac{2R}{g}}\left[\left(1+\frac{h}{R}\right)^{3/2}-1\right]$$

where R is the radius of the earth. 7

b) Deduce the differential equation of a central orbit in two-dimensional polar coordinates (r, θ). 8

14. a) A small meteor of mass m, falls into the sun when the earth is at the end of minor axis of its orbit. If M be the mass of the sun, show that the major axis of the earth's orbit is lessened by $2a\frac{m}{M}$, that the periodic time is lessened by $\frac{2m}{M}$ of a year, and that the major axis of its orbit is turned through an angle $\frac{b}{ae}\frac{m}{M}$. 8

b) A gun of mass M fires a shell of mass m horizontally and the energy of explosion is such as would be sufficient to project the shell vertically to a height h. Show that the velocity of recoil of the gun is

$$\left\{\frac{2mE}{M(m+M)}\right\}^{1/2}.$$ 7

West Bengal State University

B.A./B.Sc./B.Com. (Honours, Major, General) Examinations, 2014

PART – III

MATHEMATICS – Honours

PAPER –V

Duration : 4 Hours Maximum Marks : 100

Candidates are required to give their answers in their own words as far as practicable.

The figures in the margin indicate full marks.

GROUP - A

(Marks : 70)

Answer question No. 1 and any *five* from the rest.

1. Answer any *five* of the following : $5 \times 3 = 15$

a) Prove or disprove :The range of any convergent sequence in $\mathbb{R}$ is a compact set.

b) Prove that the function $f : [-1, 1] \rightarrow \mathbb{R}$ defined by $f(x) = x^4 + x^3$ is of bounded variation over $[-1, 1]$.

c) If e is defined by $\int_1^e \frac{dt}{t} = 1$, prove that $2 < e < 3$.

d) Test the convergence of $\int_1^\infty \frac{\cos ax - \cos bx}{x} dx$; $a, b \in \mathbb{R} - \{0\}$.

e) Check whether true or false : There exists no power series $\sum_{n=0}^{\infty} a_n x^n$ with radius of convergence 1 which is not convergent at both $x = \pm 1$.

f) Let $x : [0, 1] \to I\!R$ and $y : [0, 1] \to I\!R$ be defined by

$$x(t) = \frac{1}{3^n}, \ \frac{1}{3^{n+1}} < t < \frac{1}{3^n}; \ n = 0, 1, 2, ...$$
$$= 0, \ t = 0$$

and $y(t) = t^3 \sin \frac{1}{t^2}, \ t \neq 0$

$$= 0, \ t = 0$$

Prove that the curve $\gamma = (x, y)$ is rectifiable.

g) Determine the perimeter of the cardioid $r = a(1 + \cos\theta)$, $a > 0$.

h) Evaluate $\iint \sqrt{4a^2 - x^2 - y^2}\ dx\, dy$ taken over the upper half of the circle $x^2 + y^2 - 2ax = 0$.

i) Prove or disprove : The trigonometric series $\sum_{n=1}^{\infty} \frac{\sin nx}{n^2}$ represents a Fourier series.

2\. a) Prove that a compact subset of $I\!R$ is closed and bounded in $I\!R$. 5

b) Discuss whether the set $\left\{ 1 + \frac{(-1)^n}{n} \right\}$ is compact or not. 3

c) Every point of a compact set S in $I\!R$ is an isolated point of S. Prove that S is finite. 3

3\. a) State and prove Taylor's theorem for a real-valued function of two independent variables. 1 + 3

b) If $f(x, y) = \sin \pi x + \cos \pi y$, use Mean value theorem to express $f\left(\frac{1}{2}, 0\right) - f\left(0, -\frac{1}{2}\right)$ in terms of the first order partial derivatives of f and deduce that there exists θ in $(0, 1)$ such that

$$\frac{4}{\pi} = \cos\frac{\pi}{2}\theta + \sin\frac{\pi}{2}(1 - \theta).$$ 3

c) Prove that the volume of the greatest rectangular parallelopiped that can be inscribed in the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ is $\frac{8\,abc}{3\sqrt{3}}$. 4

4. a) Let $f : [\, a, b \,] \to I\!R$ be a bounded function on $[\, a, b \,]$. Prove that to each positive ε there corresponds a positive δ such that $\underline{\int_a^b} f - \varepsilon < L\,(P, f)$ for all partitions P of $[\, a, b \,]$ satisfying $\|P\| \le \delta$.

Further if $\{P_n\}$ be a sequence of partitions of $[\, a, b \,]$ such that $\|P_n\| \to 0$ as $n \to \infty$ then prove that

$$\lim_{n \to \infty} L\left(P_n, f\right) = \underline{\int_a^b} f.$$ 3 + 2

b) Prove that $f : [\, 0, 1 \,] \to I\!R$ defined by $f\,(\,0\,) = 0$ and $f\,(\,x\,) = \left[\, x^{-1} \right]^{-1}$, for $0 < x \le 1$ is integrable, where $[\, y \,]$ denotes the integral part of y. 2

c) A function $f : [\, 0, 1 \,] \to I\!R$ is defined by

$$f\,(x) = x + x^3, \text{ if } x \text{ is rational}$$
$$= x^2 + x^3, \text{ if } x \text{ is irrational.}$$

Evaluating $\underline{\int_0^1} f$ and $\overline{\int_0^1} f$, examine the integrability of f in $[\, 0, 1 \,]$. 4

5. a) If $f : [\, a, b \,] \to I\!R$ be integrable on $[\, a, b \,]$ then prove that the function $F : [\, a, b \,] \to I\!R$ defined by $F\,(\,x\,) = \int_a^x f\,(t)\, dt$ is continuous on $[\, a, b \,]$.

Give an example to show that continuity of f is not needed for the continuity of F. 3 + 2

b) Prove that for $0 < c < 1$, $2\int_0^1 \sqrt{1 - c^2 \sin^2 x}\, dx > \sqrt{1 - c^2} + 1$. 3

c) Let $f(x) = x[x]$, $x \in [0, 3]$. Show that f is integrable on $[0, 3]$. Further show that evaluation of $\int_0^3 f$, cannot be done by the fundamental theorem of Integral calculus. 3

a) Show that the integral $\int_0^1 x^{p-1} \log x\, dx$ is convergent if and only if $p > 0$. 4

b) Test the convergence of the integral $\int_0^1 \sin x^{1/3} \frac{\cos \sqrt{x}}{\sqrt{x}} dx$. 3

c) If f is continuous on $[0, 1]$, show that $\int_0^1 \frac{f(x)}{\sqrt{1 - x^2}} dx$ is convergent. 4

a) Define uniform convergence of a sequence of functions $\{f_n\}_n$ on $S\ (\subset \mathbb{R})$.

A sequence of functions $\{f_n\}$ is uniformly convergent on $S\ (\subset \mathbb{R})$ to a function f and $\lim_{x \to x_0} f_n(x) = a_n$, where $x_0 \in S'$. Prove that the sequence $\{a_n\}$ is convergent and $\lim_{x \to x_0} f(x)$ exists and is equal to $\lim_{n \to \infty} a_n$. 1 + 4

b) For a sequence of functions $\{f_n\}$, where each f_n is differentiable on $[a, b]$, does uniform convergence of $\{f_n\}$ implies uniform convergence of $\{f_n'\}$? Justify your answer. 3

c) Examine uniform convergence of $\{f_n\}$ on $[0, 1]$, where $f_n(x) = \frac{n^2 x}{1 + n^4 x^2}$, $x \in [0, 1]$, for each $n \in \mathbb{N}$. 3

8. a) Prove that the series $\sum_{n=0}^{\infty} (-1)^n \frac{\cos nx}{p^2 - n^2}$ is uniformly convergent on any closed and bounded interval $[a, b]$, where $p \neq 0, \pm 1, \pm 2, \ldots$. 4

b) Let $\sum_{1}^{\infty} f_n(x)$ be uniformly convergent to $f(x)$ on $[a, b]$, where each f_n is continuous on $[a, b]$ and let $g : [a, b] \to \mathbb{R}$ be integrable on $[a, b]$; prove that $\int_a^b f(x)\, g(x)\, dx = \sum_{1}^{\infty} \int_a^b f_n(x)\, g(x)\, dx$. 3

c) A series $\sum_{n=1}^{\infty} f_n(x)$ of differentiable functions f_n on $[0, 1]$ is such that

$$S_n(x) = \sum_{i=1}^{n} f_i(x) = \frac{\log\left(1 + n^4 x^2\right)}{2n^2},\ x \in [0, 1] \text{ and } n \in N.$$ Show that

$$\frac{d}{dx} \sum_{n=1}^{\infty} f_n(x) = \sum_{n=1}^{\infty} f_n'(x),\ x \in [0, 1].$$

What can be said about uniform convergence of $\sum_{1}^{\infty} f_n'(x),\ x \in [0, 1]$? 4

9. a) If $\sum_{0}^{\infty} a_n x^n$ be a power series with radius of convergence $R\ (>0)$ and $\sum_{0}^{\infty} a_n R^n$ is convergent then prove that $\sum_{0}^{n} a_n x^n$ is uniformly convergent on $[0, R]$.

Further if the sum of $\sum_{0}^{\infty} a_n x^n$ be $f(x)$ on $(-R, R)$, prove that

$$\sum_{n=0}^{\infty} a_n R^n = \lim_{x \to R-} f(x).$$ 4 + 2

b) From the relation $\sin^{-1} x = \int_0^x \frac{dx}{\sqrt{1-x^2}}, |x| < 1$; show that

$\sin^{-1} x = x + \frac{1}{2} \cdot \frac{x^3}{3} + \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{x^5}{5} + \ldots, |x| \le 1$. Hence deduce that

$\frac{\pi}{2} = 1 + \frac{1}{2} \cdot \frac{1}{3} + \frac{1.3}{2.4} \cdot \frac{1}{5} + \ldots$ 4 + 1

10. a) Obtain the Fourier series of the function $f : [-\pi, \pi] \to \mathbb{R}$ defined as

$f(x) = \cos x, 0 \le x \le \pi$

$= -\cos x, -\pi \le x < 0.$

Hence find the sum of the series $\frac{2}{1.3} - \frac{6}{5.7} + \frac{10}{9.11} - \ldots$. 3 + 1

b) Show that $\int_0^{\pi/2} \log \left(\frac{a + b \sin \theta}{a - b \sin \theta} \right) \cdot \frac{1}{\sin \theta} \, d\theta = \pi \sin^{-1} \frac{b}{a}; a > b \ge 0.$ 4

c) Show that $\iint_E y^2 \sqrt{a^2 - x^2} \, dx \, dy = \frac{32}{45} a^5$, where

$E = \{ (x, y) \in \mathbb{R}^2 : x^2 + y^2 \le a^2 \}.$ 3

GROUP - B

(Marks : 15)

Answer any *one* of the following.

11. a) Let $C[0, 1]$ denotes the set of all real valued continuous functions on $[0, 1]$. For $x, y \in [0, 1]$, let $d(x, y) = \int_0^1 |x(t) - y(t)| \, dt$. Show that d is a metric on $C[0, 1]$. 4

b) A subset of a metric space (X, d) is said to be closed if $X - A$ is an open set. Show that an arbitrary union of open sets is an open set and an arbitrary intersection of closed sets is a closed set. 3 + 3

c) If $\{x_n\}, \{y_n\}$ are convergent sequences in a metric space (X, d), show that $\lim_{n \to \infty} d(x_n, y_n)$ exists.

Hence correct or justify : $d(x, y)$ is a continuous function. 3 + 2

12. a) Let X denotes the set of all real valued sequences, and let $d : X \times X \to \mathbb{R}$ be defined by

$$d(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|x_n - y_n|}{1 + |x_n - y_n|}, \quad \text{where} \quad x = \{x_n\} \quad \text{and} \quad y = \{y_n\}.$$

Show that d is a metric. 4

b) Define Cauchy sequence in a metric space (X, d). Prove that a Cauchy sequence in a metric space is convergent *iff* it has a convergent subsequence. Define the term 'complete metric space'. 1 + 4 + 1

c) Let $C[a, b]$ denotes the metric space of all continuous functions defined on closed bounded interval $[a, b]$ with the usual metric $d(x, y) = \sup_{a \le t \le b} |x(t) - y(t)|$.

Show that in this metric space a sequence $\{x_n\}$ converges to x *iff* $\{x_n(t)\}$ converges uniformly to $x(t)$ on $[a, b]$. 5

GROUP - C

(Marks : 15)

Answer any *one* of the following.

13. a) If Z is a point on the complex plane and (α, β, γ) is the stereographic projection on the Riemann sphere $x^2 + y^2 + \left(Z - \frac{1}{2}\right)^2 = \frac{1}{4}$ then prove that $Z = \frac{\alpha + i\beta}{1 - \gamma} \left(i = \sqrt{-1}\right)$. 5

b) Let u, v be real-valued functions such that $f(x+iy) = u(x, y) + iv(x, y)$ is differentiable at $Z_0 = x_0 + iy_0$. Then prove that the function u and v are differentiable at the point (x_0, y_0) and satisfy the Cauchy-Riemann equation. 5

c) Prove that $u(x, y) = x^3 - 3xy^2$ $[x, y \in R]$ is a harmonic function on $\mathbb{R}^2$. Find the harmonic conjugate of u. 5

a) Prove that the function

$$f(x+iy) = \begin{cases} \dfrac{x^3(1+i) - y^3(1-i)}{x^2+y^2}, & \text{if } x^2+y^2 \neq 0 \\ 0, & \text{if } x^2+y^2 = 0 \end{cases}$$

Satisfies Cauchy-Riemann equations at the origin but $f'(0)$ does not exist. 5

b) Find the points where the following function f is differentiable and hence, deduce that it is nowhere analytic : $f = u + iv$, where $u(x, y) = x^2y^2$, $v(x, y) = 2x^2y^2$. 5

c) Prove that a function $f : D \to \mathbb{C}$, $D \subset \mathbb{C}$ is continuous at $Z_0 \in D$ if and only if $f(Z_0) = \lim_{n \to \infty} f(Z_n)$ whenever $Z_n \to Z_0$ as $x \to \infty$, $Z_n \in D$, for $n \in \mathbb{N}$. 5
