West Bengal State University

B.A./B.Sc./B.Com. (Honours, Major, General) Examinations, 2014

PART – II

MATHEMATICS – (Honours)

PAPER – III

Duration : 4 Hours

Maximum Marks : 100

Candidates are required to give their answers in their own words as far as practicable.

The figures in the margin indicate full marks.

GROUP - A

Answer any *three* questions. $3 \times 5 = 15$

1. Reduce the reciprocal equation $6x^6 - 25x^5 + 31x^4 - 31x^2 + 25x - 6 = 0$ to the standard form and solve it.

2. Solve $x^3 + 12x - 12 = 0$ by Cardan's method.

3. Solve $2x^4 + 6x^3 - 3x^2 + 2 = 0$ by Ferrari's method.

4. If α is an imaginary root of $x^7 - 1 = 0$, find the equation whose roots are $\alpha + \alpha^6$, $\alpha^2 + \alpha^5$ and $\alpha^3 + \alpha^4$.

5. Let a, b, c, d be four positive real numbers such that $a + b + c + d = p$ and $abcd = q$, where p and q are constants. Find the least and greatest values of $(p - a)(p - b)(p - c)(p - d)$.

6. a) Let x and y be positive numbers such that $12x^3y^4 = 1$. Find the least value of $2x + 3y$. 2

b) If a, b and c be positive real numbers such that $a + b + c = 1$, then show that $\left(a + \frac{1}{a}\right)^2 + \left(b + \frac{1}{b}\right)^2 + \left(c + \frac{1}{c}\right)^2 \geq \frac{100}{3}$. 3

GROUP - B

Answer any *one* question. 1 × 10 = 10

7. a) Prove that every proper subgroup of a group of order 6 is cyclic. 3

b) Let (G, o) be a group and H be a subgroup of G. For any $a, b \in G$, prove that $aH = bH$ if and only if $a^{-1} o\, b \in H$. 4

c) Express the permutation $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 9 & 7 & 5 & 4 & 3 & 6 & 1 & 8 \end{pmatrix}$ as a product of transpositions and hence determine whether σ is an even permutation. 2 + 1

8. a) Show that a finite group of order n in cyclic if it contains an element of order n. 3

b) Let (G, o) be a finite group and H be a subgroup of G. Prove that order of H is a factor of the order of G. 4

c) Let S_3 denotes the group of permutations of $\{1, 2, 3\}$. Show that S_3 is not a commutative group. Give example of a cyclic subgroup of order 3 of S_3, say H and write all the cosets of H in S_3. 1 + 1 + 1

GROUP - C

Answer any *two* questions. $2 \times 10 = 20$

9. a) Prove that the intersection of two subspaces of a vector space V over a field F is a subspace of V, but the union of two subspaces of V is not, in general, a subspace of V. 2 + 3

b) Let $\{ \alpha, \beta, \gamma \}$ be a basis of a vector space V. Show that $\{\alpha + \beta, \beta + \gamma, \gamma + \alpha\}$ is also a basis of V. 2

c) Is $w = \{ (x, 2y, 3z) \mid x, y, z \in \mathbb{R} \}$ a subspace of the vectorspace $\mathbb{R}^3$? Justify your answer. 3

10. a) Find bases for the row space and column space of $\begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & -1 \\ 1 & 1 & 1 \\ -1 & 0 & -3 \end{pmatrix}$. 2 + 2

b) Find for what values of a and b, the following system of equations

$$\begin{aligned} x + 4y + 2z &= 1 \\ 2x + 7y + 5z &= 2b \\ 4x + ay + 10z &= 2b + 1 \end{aligned}$$

has

i) no solution

ii) unique solution

iii) infinite number of solutions. 6

a) Use Cayley-Hamilton theorem to express A^{-1} as a polynomial in A and then compute A^{-1}, where $A = \begin{pmatrix} 0 & 0 & 1 \\ 3 & 1 & 0 \\ -2 & 1 & 4 \end{pmatrix}$. 2 + 3

b) Diagonalize the symmetric matrix $A = \begin{pmatrix} 6 & 4 & -2 \\ 4 & 12 & -4 \\ -2 & -4 & 13 \end{pmatrix}$ 5

a) Let $(V, \langle , \rangle)$ be a Euclidean space and let $\| \ \|$ be the induced norm on V. Prove that for any two vectors α and β of V, 3 + 2

i) $\|\alpha + \beta\| \leq \|\alpha\| + \|\beta\|$

ii) $\|\alpha + \beta\|^2 \leq 2\|\alpha\|^2 + 2\|\beta\|^2$.

b) Apply Gram-Schmidt orthonormalization process to the set of vectors $\{(1, 1, 1), (2, -2, 1), (3, 1, 2)\}$ to obtain an orthonormal basis of $\mathbb{R}^3$ with the standard inner product. 5

GROUP - D

Answer any *two* questions. $2 \times 10 = 20$

a) Let $\{x_n\}$ be a monotonic sequence of real numbers which has a convergent subsequence $\{x_{n_k}\}$ converging to l. Show that $\{x_n\}$ converges to l. 4

b) Find the upper and lower limits of the sequence $\left\{(-1)^n + \cos\frac{n\pi}{4}\right\}$. 4

c) Let $\{x_n\}$ be a sequence of real numbers such that $\lim_{n \to \infty} x_{2n} = \lim_{n \to \infty} x_{2n+1} = l$. Show that $\lim_{n \to \infty} x_n = l$. 2

14. a) Define an absolutely convergent series. Give an example to show that a convergent series of real numbers may not be absolutely convergent. 1 + 1

b) Use Abel's test to show that the following is a convergent series : 3

$$0 - \frac{1}{2} + \frac{1}{2^2} - \frac{1}{3} + \frac{2}{3^2} - \frac{1}{4} + \frac{3}{4^2} - \ldots$$

c) Let $\{u_n\}$ be a monotone decreasing sequence of positive terms such that $\lim_{n \to \infty} u_n = 0$. Show that the series $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{u_1 + \ldots + u_n}{n}$ converges. 2

d) Evaluate $\lim_{x \to \infty} \left(\frac{ax+1}{ax-1} \right)^x$ where $a > 0$. 3

15. a) Let a monotone increasing function f be bounded above on the bounded open interval (a, b). Show that $\lim_{x \to b-} f(x)$ exists. 2

b) Let $f : [0, 1] \to I\!R$ be a continuous function taking only rational values. Show that f is a constant function on $[0, 1]$. 2

c) Let $f : I \to I\!R$ be a function defined on an interval I or $I\!R$ for which there exists $M > 0$ such that $| f(x) - f(y) | \leq M | x - y |$ for all $x, y \in I\!R$. Show that f is uniformly continuous on I. Use this fact to show that $f(x) = \sqrt{x}$ is uniformly continuous on $[1, \infty)$. 2 + 2

d) Let $y = \frac{1}{\sqrt{1+2x}}$. Prove that $(1+2x)\, y_{n+1} + (2n+1)\, y_n = 0$. 2

16. a) Obtain McLaurin's infinite series expansion of $(1+x)^n$, $|x| < 1$, where $n \in I\!R - I\!N$. 4

b) Use Rolle's theorem to show that between any two distinct real roots of $e^x \cos x + 1 = 0$, ther is at leat one real root of $e^x \sin x + 1 = 0$. 2

c) Prove that $0 < \frac{1}{x} . \log \frac{e^x - 1}{x} < 1$. 2

d) Study the extreme points of $f(x) = x^x$, $x > 0$. 2

GROUP - E

Answer any *five* questions. $5 \times 5 = 25$

17. Let $f(x, y) = \begin{cases} \dfrac{x^4 + y^4}{x - y}, & x \neq y \\ 0, & x = y \end{cases}$

Show that f_x and f_y exist at (0, 0) but f is not continuous at (0, 0). State a set of sufficient conditions for the continuity of the function $f(x, y)$ at an interior point of its domain of definition. 1 + 1 + 1 + 2

18. If $f(x, y)$ be a function of two variables x and y where $x = u^2 v$, $y = v^2 u$ then show that

$$2x^2 \frac{\partial^2 f}{\partial x^2} + 2y^2 \frac{\partial^2 f}{\partial y^2} + 5xy \frac{\partial^2 f}{\partial x \, \partial y} = uv \frac{\partial^2 f}{\partial u \, \partial v} - \frac{2}{3}\left(u \frac{\partial f}{\partial u} + v \frac{\partial f}{\partial v}\right).$$ 5

19. If u, v are two polynomials in x, y and are homogeneous of degree n, prove that $u\,dv - v\,du = \dfrac{1}{n} \dfrac{\partial(u, v)}{\partial(x, y)} (x\,dy - y\,dx)$. 5

20. Let (a, b) be an interior point of domain of a function f of two variables. If f_x and f_y be both differentiable at (a, b), prove that $f_{xy}(a, b) = f_{yx}(a, b)$. 5

21. If v is a function of two variables x and y and $x = r\cos\theta$, $y = r\sin\theta$, prove that $\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2} = \dfrac{\partial^2 v}{\partial r^2} + \dfrac{1}{r^2} \dfrac{\partial^2 v}{\partial \theta^2} + \dfrac{1}{r} \dfrac{\partial v}{\partial r}$. 5

22. Prove that the following three functions u, v, w are functionally related and find the relation connecting them where,

$$u = \frac{x}{y - z}, \quad v = \frac{y}{z - x}, \quad w = \frac{z}{x - y}.$$ 5

23. Show that the functions f defined by

$$f(x, y) = \begin{cases} \dfrac{xy^2}{x^2 + y^4}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

possesses first order partial derivatives at (0, 0) yet it is not differentiable at (0, 0). 5

24. If u, v, w are the roots of the equation $(\lambda - x)^3 + (\lambda - y)^3 + (\lambda - z)^3 = 0$ in λ, prove that $\dfrac{\partial(u, v, w)}{\partial(x, y, z)} = -\dfrac{2(y - z)(z - x)(x - y)}{(v - w)(w - u)(u - v)}$ 5

25. State the conditions under which $f(x, y) = 0$ will determine y uniquely as a function of x near the point (a, b). Show that $y^2 - yx^2 - 2x^5 = 0$ determines y uniquely as a function of x near the point (1, – 1) and find $\dfrac{dy}{dx}$ at (1, – 1). 5

GROUP - F

Answer any *two* questions. $2 \times 5 = 10$

26. Find the area included between the curve $x^2 y^2 = a^2 \left(y^2 - x^2\right)$, $a > 0$ and its asymptotes.

27. Find the centre of gravity of the planar region bounded by the parabola $\sqrt{\dfrac{x}{a}} + \sqrt{\dfrac{y}{b}} = 1$ and the coordinate axes.

28. Find the volume of the solid bounded by the surface generated by revolving the cissoid $y^2 = \frac{x^3}{2a - x}$, $a > 0$ about its asymptote.

29. Show that the moment of inertia of a thin circular ring of mass M whose outer and inner radii are a and b respectively about an axis through the centre perpendicular to the plane of the ring is $\frac{1}{2} M \left(a^2 + b^2 \right)$.
