West Bengal State University

B.A./B.Sc./B.Com. (Honours, Major, General) Examinations, 2013

PART – II

MATHEMATICS — HONOURS

Paper – III

Duration : 4 Hours] [Full Marks : 100

The figures in the margin indicate full marks.

GROUP – A

Answer any *three* questions. 3 × 5 = 15

1. Solve the equation $x^5 - 6x^4 + 7x^3 - 7x^2 + 6x - 1 = 0$.

2. Solve the equation $x^3 - 6x - 4 = 0$ by Cardan's method.

3. Solve $x^4 - 2x^3 - 5x^2 + 10x - 3 = 0$ by Ferrari's method.

4. If α is a special root of $x^{11} - 1 = 0$, prove that $(\alpha + 1)(\alpha^2 + 1)\ldots(\alpha^{10} + 1) = 1$.

5. a) Let *a, b, c* be three positive real numbers. Prove that unless *a* = *b* = *c*,

$$\frac{2}{b+c} + \frac{2}{c+a} + \frac{2}{a+b} > \frac{9}{a+b+c}.$$ 3

b) Let *a, b, c* be three positive real numbers such that *abc* = 1. Prove that $(1+a)(1+b)(1+c) \geq 8$. 2

6. a) If *a* and *b* be positive real numbers such that *a* + *b* = 1, then show that $\left(a + \frac{1}{a}\right)^2 + \left(b + \frac{1}{b}\right)^2 \geq \frac{25}{2}$. 3

b) If *x, y, z* be three positive real numbers such that $x^2 + y^2 + z^2 = 27$, then show that $x^3 + y^3 + z^3 \geq 81$. 2

GROUP – B

Answer any *one* question. 1 × 10 = 10

7. a) Prove that a cyclic group is always abelian. Is the converse true ? Justify your answer. Also give an example of an infinite cyclic group, mentioning all its generators. 2 + 1 + 2

b) Let *H* be a subgroup of *G* and *a, b* ∈ *G*. Show that either $aH \cap bH = \theta$ or $aH = bH$. 2

c) Let A_3 denote the group of even permutations of { 1, 2, 3 }. What is the order of A_3? Show that A_3 is a cyclic group. Write all the generators of A_3. 1 + 1 + 1

8. a) Prove that any two left cosets of H in a group have the same cardinality. 4

b) Prove that a group of prime order has no non-trivial subgroup. 3

c) Let $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix}$ and $\tau = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}$. Show that $(\sigma\tau)^{-1} = \tau^{-1}\sigma^{-1}$. 3

GROUP - C

Answer any *two* questions. $2 \times 10 = 20$

9. a) Prove that every finite dimensional vector space has a basis. 5

b) Let $W = \{(x, y, z) \in \mathbb{R}^3 : x - 4y + 3z = 0\}$. Show that W is a subspace of $\mathbb{R}^3$. Find a basis of W. 2 + 3

10. a) Find row rank and column rank of the matrix $\begin{pmatrix} 2 & 1 & 4 & 3 \\ 3 & 2 & 6 & 9 \\ 1 & 1 & 2 & 6 \end{pmatrix}$. Are they equal ? Justify. 5

b) Find for what values of a and b, the following system of equations has (i) no solution, (ii) unique solution, (iii) infinite number of solutions : 5

$$x + y + z = 1$$

$$x + 2y - z = b$$

$$5x + 7y + az = b^2$$

11. a) Show that the eigenvalues of a real symmetric matrix are all real. 5

b) Diagonalize the symmetric matrix : $A = \begin{pmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{pmatrix}$ 5

12. a) State and prove Cauchy-Schwarz inequality in a Euclidean space. 2 + 3

b) Apply Gram-Schmidt orthonormalization process to the set of vectors $\{(1,-1,1),(2,0,1),(0,1,1)\}$ to obtain an orthonormal basis of $\mathbb{R}^3$ with the standard inner product. 5

GROUP - D

Answer any *two* questions. $2 \times 10 = 20$

13. a) Find all the cluster points of the sequence $\{a_n\}$ where $a_n = \left(1 - \frac{1}{n^2}\right) \sin \frac{n\pi}{2}$. Hence find the upper limit and the lower limit of $\{a_n\}$. 4

b) For any bounded sequence $\{a_n\}$ of real numbers such that $\overline{\lim}\, a_n$ is finite, show that $\underline{\lim}(-a_n) = -\overline{\lim}\, a_n$. 2

c) State and prove Bolzano-Weierstrass theorem. 4

14. a) Let $\{a_n\}$ be a monotone decreasing sequence of positive real numbers so th $\lim_{n\to\infty} a_n = 0$. Then show that the series $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$ converges.

b) Examine the convergence of the series $\sum_{n=1}^{\infty}\frac{1^2.3^2.5^2...(2n-1)^2}{2^2.4^2.6^2...(2n)^2}x^{n-1}, x>0$.

c) Let $f: I \to \mathbb{R}$ be a function continuous at $c \in I$, where I is an open interval $\mathbb{R}$. Let f takes both positive and negative values in each neighbourhood of Show that $f(c)=0$.

15. a) Let $f:[a,b] \to \mathbb{R}$ be a continuous function. Then show that f is uniform continuous. Also show that the function $f(x)=\frac{1}{x}$ is not uniformly continuo on (0, 1). 3 +

b) State Rolle's theorem. Let f be a continuous function on [a, b], differentiable (a , b) such that $f(a)=f(b)=0$. Let k be a given real number. Use Rolle theorem to show that there is a $c \in (a,b)$ such that $f'(c)+kf(c)=0$. 1 +

c) If a_n is a positive monotonic decreasing function and if $\sum u_n$ is a converge series, prove that $\sum a_n u_n$ is also convergent.

16. a) Obtain Maclaurin's infinite series expansion of $\log(1+x)$ over t interval $(-1, 1]$.

b) State Cauchy's Mean Value theorem. Deduce Lagrange's Mean Value theore from Cauchy's Mean Value theorem. Let f be a continuous function defined [0, 1] which is differentiable on (0, 1). Use Cauchy's Mean Value theorem show that $f(1)-f(0)=\frac{f'(x)}{2x}$ has at least one solution in (0, 1). 1 + 1 +

c) Examine whether $f(x)=3+(x-3)^{\frac{2}{3}}$ has an extreme point at $x = 3$.

GROUP – E

Answer any *five* questions. 5 × 5 =

17. Let $B=\{(a,0)\in \mathbb{R}^2 : a \in \mathbb{R}\}$. Show that B is a closed set but not an open set in $\mathbb{R}^2$.

18. Define $f(x,y)=\begin{cases} x\sin\frac{1}{y}+y\sin\frac{1}{x}; & xy \neq 0 \\ 0, & xy=0 \end{cases}$

Show that $\lim_{(x,y)\to(0,0)} f(x,y)$ exists but the repeated limits do not exist.

19. Define $f(x,y)=\begin{cases}(ax+by)\sin\dfrac{x}{y};\ y\neq 0\\ 0, \qquad\qquad y=0\end{cases}$. Is f continuous at (0, 0) ?

20. Prove that $f(x,y)=\sqrt{|xy|}$ possesses fist order partial derivatives at (0, 0) but is not differentiable at (0, 0)

21. Let $f(x,y)=\begin{cases}xy\ ;\ |x|\geq|y|\\ -xy\ ;\ |x|<|y|\end{cases}$. Show that $f_{xy}(0,0)\neq f_{yx}(0,0)$. Which condition of Schwarz' theorem is not satisfied by f ?

22. Transform the equation $y\dfrac{\partial z}{\partial x}-x\dfrac{\partial z}{\partial y}=(y-x)z$ by introducing new independent variables $u=x^2+y^2, v=\dfrac{1}{x}-\dfrac{1}{y}$ and the new function $w=\log z-(x+y)$.

23. You are given a differentiable function $f(x,y)$. Prove that if the variables x and y are replaced by homogeneous linear functions $X=X(x,y), Y=Y(x,y)$ of x and y, then the obtained function $F(X,Y)$ is related with the given function as follows :

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=X\frac{\partial F}{\partial X}+Y\frac{\partial F}{\partial Y}$$

24. Justify the existence and uniqueness of the implicit function $y=y(x)$ for the functional equation $y^3\cos x+y^2\sin^2 x=7$ near the point $\left(\dfrac{\pi}{3},2\right)$. Also find $\dfrac{dy}{dx}\left(\dfrac{\pi}{3},2\right)$. 4 + 1

25. Show that the functions $u=3x+2y-z$, $v=x-2y+z$ and $w=x(x+2y-z)$ are dependent and find the relation between them. 2 + 3

GROUP – F

Answer any *two* questions. 2 × 5 = 10

26. Show that the area bounded by the semicubical parabola $y^2=ax^3$ and a double ordinate is $\dfrac{2}{5}$ of the area of the rectangle formed by this ordinate and the abscissa.

27. Find the coordinates of the centre of gravity of the first arc of the cycloid $x=a(t-\sin t), y=a(1-\cos t)$.

28. If the loop of the curve $2ay^2=x(x-a)^2$ revolves about the line $y=a$, then using Pappus theorem, find the volume of the solid generated.

29. Show that the moment of inertia of a truncated cone about its axis is $\dfrac{3M(a^5-b^5)}{10(a^3-b^3)}$ where a and b are the radii of the two ends and M is the mass of the truncated cone.