West Bengal State University

B.A./B.Sc./B.Com. (Honours, Major, General) Examinations, 2013

PART-I

MATHEMATICS - Honours

Paper– II

Duration : 4 Hours Full Marks : 100

Candidates are required to give their answers in their own words as far as practicable.

The figures in the margin indicate full marks.

GROUP – A

(Marks : 25)

Answer any *five* questions. 5 × 5 = 25

1. a) If $x, y \in I\!R$ and $x > 0$, $y > 0$ then prove that there exists a natural number n such that $ny > x$.

   b) Let T be a bounded subset of $I\!R$. If $S = \{ \, |x - y| : x, y \in T \}$, show that $Sup\ S = Sup\ T - Inf\ T$. 3 + 2

2. a) State Cauchy's second limit theorem. Use it to prove that

   $$\lim_{n \to \infty} \frac{\{ (n+1)(n+2) \ldots (2n) \}^{\frac{1}{n}}}{n} = \frac{4}{e}.$$

   b) Prove that the sequence $\{ x_n \}$ where

   $$x_n = \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \ldots + \frac{1}{\sqrt{n^2+n}}$$ converges to 1. 1 + 2 + 2

3. a) Prove that the sequence $\{ x_n \}$ where $x_n = \left(1 + \frac{1}{n}\right)^n$ is convergent.

   b) Prove that a non-decreasing sequence which is not bounded above diverges to ∞. 3 + 2

4. a) State Cauchy's general principle of convergence and use it to prove that sequence $\left\{\frac{n-1}{n+1}\right\}$ is convergent.

   b) Prove or disprove : Every bounded sequence is a Cauchy sequence. 1 + 2

5. a) Prove that a subset of a denumerable set is either finite or denumerable.

   b) Prove that the closed interval [a, b] is not denumerable. 2

6. Prove that every infinite bounded subset of $\mathbb{R}$ has at least one limit point in $\mathbb{R}$.

7. a) Prove that derived set of a set is a closed set.

   b) Which one of the following sets is closed and why ?

   (i) $T = \left\{ \frac{1}{n} : n \in N \right\}$

   (ii) $T \cup \{ 0 \}$. 3 +

8. a) If $\lim_{x \to a} f(x) = l \ (\neq 0)$ then prove that there exists a *nbd* of a where $f(x)$ and will have the same sign.

   b) Prove that $\lim_{x \to 0} \frac{\sin x}{x} = 1$. 3 +

9. a) If $f : [0,1] \to \{0,1\}$ be defined as

$$f(x) = \begin{cases} 1 \text{ when } x \text{ is rational} \\ 0 \text{ when } x \text{ is irrational,} \end{cases}$$

show that f is not continuous on [0, 1].

b) Is the function f defined as follows piecewise continuous ? If so, find the intervals of continuity of f.

$$f(x) = \begin{cases} 4 - x & \text{when } 0 \le x < 1 \\ 4 & \text{when } x = 1 \\ 6 - x & \text{when } 1 < x \le 2 \end{cases}$$ 3 + 2

GROUP – B

(Marks : 20)

10. Answer any *two* questions : 2 × 4 = 8

a) If $I_n = \int \dfrac{dx}{(x^2 + a^2)^n}$, then prove that

$$I_n = \frac{1}{2(n-1)\,a^2} \cdot \frac{x}{(x^2 + a^2)^{n-1}} + \frac{2n-3}{2(n-1)a^2} I_{n-1}, n\ (\ne 1) \text{ being a positive integer.}$$ 4

b) Show that $\Gamma\left(n + \dfrac{1}{2}\right) = \dfrac{\Gamma\,(2n+1)\,\sqrt{\pi}}{2^{2n}\ \Gamma\,(n+1)}$, $n > 0$ being an integer. 4

c) Prove that $B(m, n) = \displaystyle\int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}}\, dx\ (m > 0,\ n > 0)$. 4

11. Answer any *three* questions : 3 × 4 = 12

a) Show that the pedal of the circle $r = 2a\cos\theta$ with respect to the origin is the cardioid $r = a\,(1 + \cos\theta)$. 4

b) If ρ_1 and ρ_2 are radii of curvatures at two extremities of any chord of the cardioid $r = a\,(1 + \cos\theta)$ passing through the pole, prove that

$$\rho_1{}^2 + \rho_2{}^2 = \frac{16a^2}{9}.$$ 4

c) Find the asymptotes of $x^4 - 5x^2y^2 + 4y^4 + x^2 - 2y^2 + 2x + y + 7 = 0$. 4

d) Prove that the condition that $x\cos\alpha + y\sin\alpha = p$ should touch

$$x^m y^n = a^{m+n} \text{ is } p^{m+n} m^m n^n = (m+n)^{m+n} a^{m+n} \sin^n\alpha\cos^m\alpha .$$

e) Show that the points of inflexion on the curve $y^2 = (x-a)^2(x-b)$ lie on the li

$3x + a = 4b$.

f) Find the evolute of the curve

$$x = a(\cos t + t\sin t),\quad y = a(\sin t - t\cos t).$$

GROUP - C

(Marks : 30)

Answer any *three* questions. 3 × 10 = 3

12. a) Examine whether the equation

$(x^2y - 2xy^2)\,dx + (3x^2y - x^3)\,dy = 0$ is exact or not and solve it.

b) Solve $xy(1 + xy^2)\dfrac{dy}{dx} = 1$ reducing Leibnitz's linear equation.

13. a) Find the complete primitive of the equation $x^2(y - px) = p^2y$ by reducing it int

Clairaut's form.

b) Solve : $(px - y)(py + x) = a^2 p$.

14. a) Solve : $\dfrac{d^2x}{dt^2} + 4t = 0$, satisfying $x = 4$, $\dfrac{dx}{dt} = 3$, when $t = 0$.

b) Solve : $D^2 - 3D + 2y = xe^{3x} + \sin 2x$, where $D = \dfrac{d}{dx}$.

15. a) Solve by method of undetermined coefficient :

$$\frac{d^2y}{dx^2} - y = e^{3x}\cos 2x - e^{2x}\sin 3x.$$ 5

b) Solve by method of variation of parameters :

$$\frac{d^2y}{dx^2} + a^2 y = \sec ax.$$ 5

16. a) Solve : $\frac{d^2y}{dx^2} + \cot x\,\frac{dy}{dx} + 4y\,\mathrm{cosec}^2 x = 0$ by changing the independent variable. 5

b) Solve by reducing normal form

$$\frac{d^2y}{dx^2} - \frac{2}{x}\frac{dy}{dx} + \left(a^2 + \frac{2}{x^2}\right)y = 0.$$ 5

17. a) Solve $(1+x^2)^2\,\frac{d^2y}{dx^2} + 2x\,(1+x^2)\,\frac{dy}{dx} + 4y = 0$ by changing the independent variable. 5

b) Solve by method of operational factors :

$$\left[(x+3)\,D^2 - (2x+7)\,D + 2\right] y = (x+3)^2 e^x.$$ 5

GROUP-D

(Marks : 25)

Answer any *five* questions. 5 × 5 = 25

18. a) Show that the following vectors are coplanar :

$(\vec{a} - 2\vec{b} + \vec{c})$, $(2\vec{a} + \vec{b} - 3\vec{c})$, $(-3\vec{a} + \vec{b} + 2\vec{c})$ where $\vec{a}, \vec{b}, \vec{c}$ are any three non-coplanar vectors. 2

b) If $\vec{a} + \vec{b} + \vec{c} = \vec{0}$ and $|\vec{a}| = 7$, $|\vec{b}| = 3$, $|\vec{c}| = 5$ find the angle between the directions of $\vec{b}$ and $\vec{c}$. 3

19. a) Find the unit vector perpendicular to both the vectors $(3\vec{i}+\vec{j}+2\vec{k})$ a

$(2\vec{i}-2\vec{j}+4\vec{k})$. Also find the angle between them. 2

b) Show by vector method that the diagonals of a rhombus are at right angles.

20. a) Show that $[\vec{a}\times\vec{b}\quad \vec{b}\times\vec{c}\quad \vec{c}\times\vec{a}] = [\vec{a}\ \vec{b}\ \vec{c}]^2$ where [ ] denotes the scal

triple product.

b) Find the volume of the tetrahedron *ABCD* by vector method with vertic

$A(1, 1, -1)$, $B(3, -2, -2)$, $C(5, 5, 3)$, $D(4, 3, 2)$.

21. Prove that for two vectors $\vec{a}$ and $\vec{b}$,

$curl\ (\vec{a}\times\vec{b}) = \vec{a}\ div\ \vec{b} - \vec{b}\ div\ \vec{a} + (\vec{b}.\nabla)\ \vec{a} - (a.\nabla)\ \vec{b}$.

22. a) If vectors $\vec{A}$ and $\vec{B}$ are irrotational then show that the vector $\vec{A}\times\vec{B}$ i

solenoidal.

b) Find the vector equation of the plane through the point $(8\vec{i}+2\vec{j}-3\vec{k})$ and

perpendicular to each of the planes $\vec{r}.(2\vec{i}-\vec{j}+2\vec{k})=0$ and

$\vec{r}.(\vec{i}+3\vec{j}-5\vec{k})=0$.

23. a) Find the equation of the tangent plane to the surface $xz^2+x^2y-z+1=0$ at th

point $(1, -3, 2)$.

b) Find the maximum value of the directional derivative of $\phi = x^2+z^2-y^2$ at th

point $(1, 3, 2)$.

24. a) Show that the vector $\frac{\vec{r}}{r^3}$ where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, $r = |\vec{r}|$ is solenoidal. 3

b) If $\vec{r} = 3t\,\hat{i} + 3t^2\,\hat{j} + 2t^3\,\hat{k}$, find the value of the box product of $\frac{d\vec{r}}{dt}, \frac{d^2\vec{r}}{dt^2}, \frac{d^3\vec{r}}{dt^3}$. 2

25. Find the radial and transverse acceleration of a particle moving in a plane curve. 5

26. Prove that $curl\,(u\,\vec{F}) = grad\,u \times \vec{F} + u\,(curl\,\vec{F})$. Hence prove that if $u\,\vec{F} = \nabla v$, where u, v are scalar fields and $\vec{F}$ is vector field, then $(\vec{F}.\,curl\,\vec{F}) = 0$. 5