West Bengal State University

B.A./B.Sc./B.Com. (Honours, Major, General) Examinations, 2013

PART – I

MATHEMATICS — HONOURS

Paper – I

Duration : 4 Hours] [Full Marks : 10

The figures in the margin indicate full marks.

GROUP - A

Answer any *five* questions. 5 × 5 = 2

1. i) Use congruence to show that $2^{5n+3}+5^{2n+3}$ is divisible by 7, for all $n \geq 1$.

   ii) Show that the number of prime integers is infinite.

2. If p is a prime integer and a is any integer such that p does not divide a then sho that $a^{p-1} \equiv 1$ (mod p).

   Using it show that $1^p + 2^p + ... + (p-1)^p \equiv 0$ (mod p) for any prime $p > 2$. 3 +

3. i) By Fermat's theorem, show that $a^{12} - b^{12}$ is divisible by 91 if a and b are both prime to 91. 3

ii) If $d = gcd\,(a, b)$ then show that $gcd\,(a^2, b^2) = d^2$. 2

4. Show that the principal value of the ratio of $(1+i)^{1-i}$ and $(1-i)^{1+i}$ is $\sin(\log 2) + i\cos(\log 2)$. 5

5. If $\tan(\alpha + i\beta) = \tan\theta + i\sec\theta$ where α, β, θ are real numbers with $0 < \theta < \pi$, then show that $e^{2\beta} = \cot\frac{\theta}{2}$ and $\alpha = n\pi + \frac{\pi}{4} + \frac{\theta}{2}$. 5

6. i) Find the general solution of $\cos h\, z = -2$. 3

ii) Expand $\sin^7\theta$ in a series of sines of multiples of θ. 2

7. i) Apply Descartes' rule of sign to find the nature of the roots of the equation $3x^4 + 12x^2 + 5x - 4 = 0$. 2

ii) Show that the equation $x^3 - 2x - 5 = 0$ has no negative real root.

8. Let α, β, γ be the roots of $x^3 + px^2 + q = 0$. Form the equation whose roots are $\frac{1}{\alpha} + \frac{1}{\beta} + 1$, $\frac{1}{\beta} + \frac{1}{\gamma} + 1$, $\frac{1}{\gamma} + \frac{1}{\alpha} + 1$. Hence find the value of $(\alpha + \beta + \alpha\beta)(\beta + \gamma + \beta\gamma)(\gamma + \alpha + \gamma\alpha)$. 5

9. i) Find the equation whose roots are the roots of the equation $x^4 - 8x^2 + 8x + 6 = 0$, each diminished by 2. 3

ii) If $\alpha, \beta, \gamma, \delta$ are roots of $x^4 + px^3 + qx^2 + rx + s = 0$, then find the values of $\sum \alpha^2\beta\gamma$ and $\sum \frac{1}{\alpha\beta}$. 2

GROUP - B

Answer any *two* questions. 2 × 10 = 20

10. i) Let A, B, C, D be non-empty sets. Prove that $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$. 3

ii) If $f : A \to B$ is a mapping and P, Q are non-empty subsets of A, then show that

$f(P \cup Q) = f(P) \cup f(Q)$.

Is $f(P \cap Q) = f(P) \cap f(Q)$? Justify your answer. 3 + 2

iii) Find a relation on the set of positive integers which is transitive but neither reflexive nor symmetric. 2

11. i) If ρ is an equivalence relation on a non-empty set A then define equivalence class $[\,a\,]$ of an element $a \in A$ determined by ρ. Show that, either $[a] \cap [b] = \phi$ or $[a] = [b]$ for any $a, b \in A$. 1 + 2

ii) Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$, for all $x \in \mathbb{R}$ and

$$g(x) = \begin{cases} x - 1, & x \geq 0 \\ -x, & x < 0 \end{cases}.$$

Find *fog* and *gof*. Is *fog* = *gof* ? 2 + 1

iii) Let G be a semigroup and for $a, b \in G$, each of the equations $ax = b$ and $ya = b$ has solutions in G. Show that G is a group. 4

12. i) If for $a \in G$, $a^3 = e$ (= identity of G) and $b \in G$ is such that $aba^{-1} = b^2$ th

find the order of b.

ii) Let $GL(2, \mathbb{R})$ denote the group of all non-singular 2×2 matrices over

Show that $S = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} : a \neq 0 \right\}$ is a sub-group of $GL(2, \mathbb{R})$. Is this sub-grou

commutative ?

iii) Let H be a sub-group of a group G. Show that $K = \{ gHg^{-1} : g \in G \}$ is a sub-grou

of G. Also show that $|H| = |K|$, where $|H|$ stands for the order of H.

13. i) If R is a ring with unity 1 then show that characteristic of R is n if and only

$n.1 = 0$.

ii) Show that, the set of integers modulo 6 forms a ring with respect to th

addition and multiplication modulo 6.

Is this an integral domain ? Justify. 3 + 1

iii) Prove that, a finite integral domain is a field. Give an example to show that the

result is false if the 'finiteness' condition is dropped. 3 + 1

GROUP - C

Answer any *three* questions. $3 \times 5 = 15$

14. If $A = \begin{pmatrix} a & b & c \\ b & c & a \\ c & a & b \end{pmatrix}$ is an orthogonal matrix, then show that $a+b+c = \pm 1$.

Evaluate det ($adj\, A$). 5

15. Find the non-singular matrices P and Q such that PAQ is in the normal form and hence find the rank of the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 3 & 1 & 1 \end{bmatrix}$. 5

16. Without expanding, prove that $\begin{vmatrix} 2bc - a^2 & c^2 & b^2 \\ c^2 & 2ca - b^2 & a^2 \\ b^2 & a^2 & 2ab - c^2 \end{vmatrix} = (a^3 + b^3 + c^3 - 3abc)^2$. 5

17. Find the values of k for which the system of equations

$x + y - z = 1$

$2x + 3y + kz = 3$

$x + ky + 3z = 2$

has (i) no solution, (ii) more than one solutions, (iii) unique solution. 5

18. Reduce the matrix $A = \begin{pmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & 2 & 0 \end{pmatrix}$ to a row-reduced Echelon form

find its rank.

19. Reduce the quadratic form $x^2 + 2y^2 + 4z^2 + 2xy - 4yz - 2zx$ to normal for

that it is positive definite.

GROUP - D

Answer any *one* question.

20. i) Show that the set of all feasible solutions of an LPP is a convex set.

ii) Use graphical method to solve the following LPP :

Minimize $Z = 2x_1 + 3x_2$

subject to $-x_1 + 2x_2 \leq 4$

$x_1 + x_2 \leq 6$

$x_1 + 3x_2 \geq 9$

and $x_1, x_2 \geq 0$

iii) How many basic solutions the following system of equations has

$2x_1 - 5x_2 + x_3 + 3x_4 = 4$

$3x_1 - 10x_2 + 2x_3 + 6x_4 = 12.$

Justify your answer and find all basic solutions.

21. i) Prove that $x_1 = 2,\ x_2 = 1$ and $x_3 = 3$ is a feasible solution of the system of equations

$$4x_1 + 2x_2 - 3x_3 = 1$$

$$-6x_1 - 4x_2 + 5x_3 = -1.$$

Reduce the feasible solution to a basic feasible solution. 1 + 4

ii) A soft drink plant has three machines M_1, M_2, M_3. It produces and sells 500 ml, 800 ml and 1 Lt. bottles. The capacities of the machines for the production of number of bottles per minute are as follows : 5

Machines \ Prod.	500 ml	800 ml	1 Lt.
M_1	100	40	20
M_2	60	75	90
M_3	40	100	120

The machines M_1, M_2, M_3 can run 8 hrs, 6 hrs and 4 hrs. respectively per day and 5 days a week. The weekly production of drinks cannot exceed 60,000 Lt. The market can absorb 20,000 bottles of 500 ml, 7000 bottles of 800 ml and 5000 bottles of 1 Lt. The profits per bottle on three types are Rs. 2, Rs. 3 and Rs. 5 respectively. The producer wishes to maximize his profit subject to all production and marketing restrictions. Formulate this as an LPP.

GROUP - E

Section - I

Answer any *three* questions. $3 \times 5 = 15$

22. Reduce the equation $x^2 + 24xy - 6y^2 + 28x + 36y + 16 = 0$ to standard form and find the centre and eccentricity of the conic represented by it.

23. If the equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents two intersecting straight lines, then show that the square of the distance of the point of intersection of the straight line from the origin is $\dfrac{c(a+b) - f^2 - g^2}{ab - h^2}$.

    Show also that the area of the parallelogram formed by them and the pair of straight lines $ax^2 + 2hxy + by^2 - 2gx - 2fy + c = 0$ is $\dfrac{2c}{\sqrt{h^2 - ab}}$.

24. A point moves so that the distance between the feet of the perpendiculars from it the straight lines given by the equation $ax^2 + 2hxy + by^2 = 0$ is a constant $2d$. S that its locus is $(x^2 + y^2)(h^2 - ab) = d^2[(a-b)^2 + 4h^2]$.

25. If the sum of the ordinates of two points on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is b, show that the locus of the pole of the chord which joins them is $b^2x^2+a^2y^2=2a^2by$. 5

26. Show that the condition that the straight line $\frac{l}{r}=a\cos\theta+b\sin\theta$ may touch the conic $\frac{l}{r}=1-e\cos\theta$ is $(al+e)^2+b^2l^2=1$. 5

Section - II

Answer any *three* questions. $3\times 5=15$

27. Show that the pair of straight lines whose direction cosines are given by $3lm-4ln+mn=0$ and $l+2m+3n=0$ are at right angle. 5

28. i) Find the equation of the plane through the point (x_1, y_1, z_1) parallel to the plane $ax+by+cz=0$. 2

ii) Find the equation of the plane which passes through the point (2, 1, −1) and is orthogonal to each of the planes $x-y+z=1$ and $3x+4y-2z=0$. 3

29. If P be the point (2, 3, −1), find the equation of the plane through P at right angle to the straight line OP where O is the origin. 5

30. Find the image of the point ($-3, 8, 4$) in the plane $6x - 3y - 2z + 1 = 0$.

31. Find the distance of the point (4, 1, 1) from the straight line $x + y + z = 4$, $x - 2y - z = 4$. Find the equation of the perpendicular and also find the foot of the perpendicular.