West Bengal State University

B.A./B.Sc./B.Com. (Honours, Major, General) Examinations, 2012

Part – II

MATHEMATICS — HONOURS

Paper - III

Duration : 4 Hours [Maximum Marks : 100

The figures in the margin indicate full marks.

GROUP – A

Answer any *three* questions. 3 × 5 = 15

1. Solve the equation :

$$2x^5 - 7x^4 - x^3 - x^2 - 7x + 2 = 0$$

2. Solve the equation by Cardan's method :

$$28x^3 - 9x^2 + 1 = 0.$$

3. Solve the equation by Ferrari's method :

$$x^4 - 9x^3 + 28x^2 - 38x + 24 = 0.$$

4. Find the special roots of the equation $x^{24} - 1 = 0$ and deduce that

$$\cos\frac{\pi}{12} = \frac{\sqrt{3}+1}{2\sqrt{2}}, \quad \cos\frac{5\pi}{12} = \frac{\sqrt{3}-1}{2\sqrt{2}}.$$

5. a) If a_1, a_2, a_3, a_4 be distinct positive numbers and $s = a_1 + a_2 + a_3 + a_4$ then show that

$$\frac{s}{s-a_1} + \frac{s}{s-a_2} + \frac{s}{s-a_3} + \frac{s}{s-a_4} > 5\frac{1}{3}.$$

b) Show that $(n+1)^n > 2^n n!$ 3 + 2

6. a) If x, y, z be positive rational numbers then show that

$$\left(\frac{x^2 + y^2 + z^2}{x + y + z} \right)^{x+y+z} \geq x^x y^y z^z .$$

b) If a, b, c be all positive and $abc = k^3$, then prove that

$$(1 + a) (1 + b) (1 + c) \geq (1 + k)^3 .$$

3 + 2

GROUP – B

Answer any *one* question. 1 × 10 = 10

7. a) Let $\equiv$ be an equivalence relation on the set $\mathbb{Z}$ of integers given by $a \equiv b$ if $b - a$ is divisible by 8. For any $a \in \mathbb{Z}$ let $[a]$ denote its equivalence class under this equivalence relation. Let $\mathbb{Z}_8$ denote the set of all equivalence classes under this equivalence relation. Let + be an operation on $\mathbb{Z}_8$ given by $[a] + [b] = [a + b]$. Show that $\mathbb{Z}_8$ is a finite cyclic group. Write all the generators of $\mathbb{Z}_8$. 3 + 2

b) Let G be a group and H be a subgroup of G. Show that any left coset of H other than H is not a subgroup of G. 2

c) Let S_3 denote the group of permutations of { 1, 2, 3 }. Write all the subgroups of S_3. 3

8. a) Prove that any subgroup of a cyclic group is cyclic. 4

b) Let G be a finite group of order n and $a \in G$. Show that n is divisible by the order of a. 3

c) Let $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 1 & 5 & 4 \end{pmatrix}$ and $\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 5 & 4 & 2 \end{pmatrix}$. Find $\sigma\tau$. Is $\sigma\tau$ an even permutation? 1 + 2

GROUP – C

Answer any *two* questions. 2 × 10

9. a) Let A and B be two subspaces of a real vector space V. Show that the set $S = \{ a + b : a \in A, b \in B \}$ is a subspace of V. Let W be a subspace of V that $A, B \subseteq W$. Show that $S \subseteq W$.

   b) Let $W = \{ (x, y, z) \in I\!R^3 : x - y + 2z = 0 \}$. Show that W is a subspace of Find a basis of W.

10. a) Let A be a $m \times p$ matrix and B be a $p \times n$ matrix over $I\!R$. Show that $AB \leq min \{ \text{rank } A, \text{rank } B \}$.

    b) Find for what values of a and b, the following system of equations has solution, (ii) unique solution, (iii) infinite number of solutions :

$$x + y + z = 6$$

$$x + 2y + 3z = 10$$

$$x + 2y + az = b$$

11. a) Let λ be a characteristic value of a real skew symmetric square matrix order n. Show that either $\lambda = 0$ or λ is purely imaginary.

    b) State and prove Cayley-Hamilton theorem for a square matrix.

12. a) Let V be a Euclidean space. Prove that for any $\alpha, \beta \in V, || \alpha + \beta || \leq || \alpha || + || \beta ||$.

    b) Define an orthogonal set of vectors in a Euclidean space. Show that orthogonal set of non-null vectors in a Euclidean space is linearly independen

    c) Apply Gram-Schmidt orthonormalization process to the set of vectors $\{ (1, 0, 1), (1, 0, -1), (1, 3, 4) \}$ to obtain an orthonormal basis of $I\!R^3$ the standard inner product.

GROUP – D

Answer any *two* questions. $2 \times 10 = 20$

13. a) What do you understand by the symbols $\overline{\lim}\ x_n$ and $\underline{\lim}\ x_n$ where $\{ x_n \}$ is a sequence of real numbers ? Show that $\{ x_n \}$ converges iff these two limits are both finite and equal. $2 + 4$

b) Prove that if a sequence $\{ x_n \}$ converges to l, then every subsequence of $\{ x_n \}$ also converges to l. 4

14. a) State Cauchy's condensation test for convergence or divergence of a series of positive terms. Show that

$$\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p} \text{ converges if } p > 1 \text{ and diverges if } p \leq 1.$$ 6

b) Show that the function $\cos \frac{1}{x}$ defined in $0 < x < 1$ is not uniformly continuous there. 4

15. a) State and prove the Inermediate Value Theorem for a function continuous in a closed interval. 5

b) A function $f : [a, b] \rightarrow R$ is differentiable on $[a, b]$ such that the derived function $f' : [a, b] \rightarrow R$ is a continuous function. Prove that $\exists$ a positive constant k such that for any pair of points $x_1, x_2 \in [a, b]$

$$| f(x_1) - f(x_2) | \leq k \, | x_2 - x_1 |.$$ 3

c) Prove that an absolutely convergent series is convergent. 2

16. a) Prove that inequality $0 < \frac{1}{\log(1+x)} - \frac{1}{x} < 1 \;(x > 0)$.

b) If $y = (x^2 - 1)^n$, prove that

$$(1 - x^2)\, y_{n+2} - 2xy_{n+1} + n(n+1)\, y_n = 0.$$

Deduce that $u = \frac{d^n}{dx^n}(x^2 - 1)^n$ is a solution of the differential equation

$$(1 - x^2)\frac{d^2u}{dx^2} - 2x\frac{du}{dx} + n(n+1)\, u = 0.$$

c) Evaluate $\lim_{x \to 0}\left(\frac{\sin x}{x}\right)^{1/x^2}$.

GROUP - E

Answer any *five* questions. 5 × 5 = 2

17. Let $S = \{(x, y) \in I\!R^2 : x, y \in Q\}$. Show that S is neither open nor closed in $I\!R^2$. 2 +

18. Define $f(x, y) = \begin{cases} \frac{xy}{x^2 + y^2} & ; \; x^2 + y^2 \neq 0 \\ 0 & ; \; x^2 + y^2 = 0 \end{cases}$

Show that $f(x, 0)$ is continuous at $x = 0$ and $f(0, y)$ is continuous at $y = 0$ bu $f(x, y)$ is not continuous at $(0, 0)$. 1 + 1 +

19. Define $f(x, y) = \begin{cases} \frac{xy}{\sqrt{x^2 + y^2}} & ; \; (x, y) \neq (0, 0) \\ 0 & ; \; (x, y) = (0, 0) \end{cases}$

Show that f is not differentiable at $(0, 0)$ though f is continuous at $(0, 0)$. 3 +

20. Let $f(x, y) = \begin{cases} (x^2 + y^2) \log (x^2 + y^2) & ; \ (x, y) \neq (0, 0) \\ 0 & ; \ (x, y) = (0, 0) \end{cases}$

Show that f does not satisfy all the conditions of Schwarz' theorem but

$f_{xy}(0, 0) = f_{yx}(0, 0)$. 3 + 2

21. If $u(x, y) = \varphi(xy) + \sqrt{xy}\ \psi\left(\frac{y}{x}\right)$, $x \neq 0$, $y \neq 0$ where φ and ψ are twice differentiable functions, prove that $x^2 \frac{\partial^2 u}{\partial x^2} - y^2 \frac{\partial^2 u}{\partial y^2} = 0$.

22. Transform the equation $x^2 \frac{\partial z}{\partial x} + y^2 \frac{\partial z}{\partial y} = z^2$ by introducing new independent variables $u = x$, $v = \frac{1}{y} - \frac{1}{x}$ and the new function $w = \frac{1}{z} + \frac{1}{x}$.

23. If $H(x, y)$ be a homogeneous function of x and y of degree n having continuous first order partial derivatives and $u = (x^2 + y^2)^{-\frac{n}{2}}$, show that

$$\frac{\partial}{\partial x}\left(H \frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(H \frac{\partial u}{\partial y}\right) = 0.$$

24. Justify the existence and uniqueness of the implicit function $y = y(x)$ for the functional equation $2xy - \log(xy) = 2e - 1$ near the point $(1, e)$. Also find

$\left.\frac{dy}{dx}\right|_{(1, e)}$. 4 + 1

25. Show that the functions $u = \frac{x}{y - z}$, $v = \frac{y}{z - x}$ and $w = \frac{z}{x - y}$ are dependent and find the relation between them. 2 + 3

GROUP - F

Answer any *two* questions. $2 \times 5 =$

26. Find the area of the portion of the circle $x^2 + y^2 = 1$ which lies inside the parabo $y^2 = 1 - x$.

27. Show that the volume of the solid obtained by revolving the cardioid $r = a(1 + \cos$ about the initial line is $\frac{8}{3}\pi a^3$.

28. Find the centroid of the area in the first quadrant bounded by $y = x^2$ and $y = x^3$.

29. Prove that the moment of inertia of a solid right circular cone of height h and sem vertical angle α about its axis is $\frac{3}{10} mh^2 \tan^2 \alpha$.