West Bengal State University

B.A./B.Sc./B.Com. (Honours, Major, General) Examinations, 2012

PART-I

MATHEMATICS - (HONOURS)

Paper– II

Duration : 4 Hours Full Marks : 100

Candidates are required to give their answers in their own words as far as practicable.

The figures in the margin indicate full marks.

GROUP – A

(Marks : 25)

Answer any *five* questions : $5 \times 5 = 25$

1. a) State the supremum property of R. Show that the supremum property is not satisfied by the set Q of rational numbers.

   b) State the density property of R. $1 + 3 + 1$

2. a) Define interior point of a subset S of R. State with reasons whether a is an interior point of $S = [\,a,\ b\,]$ or not. $1 + 1$

   b) Prove that if $S \subset R$, then int (S) is the largest open set contained in S. 3

3. a) Define isolated point of $S \subset R$. Find the isolated points of the set Q of rational numbers. $1 + 1$

   b) If $A,\ B \subset R$ then prove that $d\,(\,A \cap B\,) \subset d\,(\,A\,) \cap d\,(\,B\,)$ where $d\,(\,A\,)$ denotes derived set of A. Give an example to show that. $d\,(A \cap B) \neq d\,(A) \cap d\,(B)$ $1 + 2$

4. a) Prove that a convergent sequence is bounded. 2

   b) Is a bounded sequence always convergent ? Give reasons in support of your answer. 1

c) Prove that the sequence $\{x_n\}$ where

$x_n = 1 + \frac{1}{1!} + \frac{1}{2!} + ... + \frac{1}{n!}$ is bounded above.

Test whether it is convergent or not.

5. a) If $\underset{n \to \infty}{Lt} u_n = l$ then prove that $\underset{n \to \infty}{Lt} \frac{u_1 + u_2 + ... + u_n}{n} = l$.

b) Using this, prove that if $\underset{n \to \infty}{Lt} u_n = l$ where $u_n > 0$ for all $n \in N$ and $l \neq 0$

$\underset{n \to \infty}{Lt} \sqrt[n]{u_1 \cdot u_2 \cdots u_n} = l$

6. a) Prove that a convergent sequence is a Cauchy sequence.

b) Use Cauchy's general principle of convergence to prove that the sequence {
where $u_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$, is not convergent.

c) Prove that the sequence $\left\{\frac{1}{n}\right\}$ is a Cauchy sequence.

7. a) Prove that the union of an enumerable number of enumerable set enumerable.

b) Use it to prove that the set Q of rational numbers is enumerable.

8. a) Show that $\underset{x \to 0}{Lt} [x]$ does not exist where $[x]$ represents the greatest int not exceeding x. Which kind of discontinuity is there at $x = 0$?

b) Prove that $\underset{x \to 0}{Lt} x^2 \cos \frac{1}{x^2} = 0$ but $\underset{x \to 0}{Lt} \cos \frac{1}{x}$ does not exist.

9. a) When is a real valued function f of x, defined on $[a, b]$ said to be piece- continuous ? Is the function $f(x) = x + [x]$ piece-wise continuous in [0, 2 If so, find the intervals of continuity of f. ($[x]$ has its usual meaning.)

b) Give an example of a function which is nowhere continuous. Give reasor support of your answer.

GROUP-B

(Marks : 20)

10. Answer any *two* questions : $2 \times 4 = 8$

a) If $I_{m,n} = \int_0^{\frac{\pi}{2}} \sin^m x \cos^n x dx$, then prove that

$I_{m,\,n} = \frac{n-1}{m+n} I_{m,\,n-2}$, m, n being positive integer. Hence or otherwise show that

$I_{m,\,n} = \frac{1\,.\,3\,.\,5\,.\,\cdots(m-1)\,.\,1\,.\,3\,.\,5\,.\,\cdots(n-1)}{2\,.\,4\,.\,6\,.\,\cdots\,(m+n)}\frac{\pi}{2}$, when both m and n are positive integers. $3 + 1$

b) Show that $\beta(m, n) = \int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}} dx \;\; (m > 0,\ n > 0)$. 4

c) Prove that $\Gamma(m)\,\Gamma\left(m + \frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2m-1}} \Gamma(2m)$, m being an integer. 4

11. Answer any *three* questions : $3 \times 4 = 12$

a) Show that the pedal of the curve $x^m y^n = a^{m+n}$ with respect to origin is $r^{m+n} = a^{m+n} \frac{(m+n)^{m+n}}{m^m n^n} \cos^m \theta \sin^n \theta$. 4

b) Find the asymptotes of

$$x^2(x^2 - y^2)(x - y) + 2x^3(x - y) - 4y^3 = 0\ .$$ 4

c) If $x \cos\alpha + y \sin\alpha = p$ touches the curve

$$\frac{x^m}{a^m} + \frac{y^m}{b^m} = 1,$$

then show that $(a\cos\alpha)^{\frac{m}{m-1}} + (b\sin\alpha)^{\frac{m}{m-1}} = p^{\frac{m}{m-1}}$. 4

d) If ρ_1 and ρ_2 be the radii of curvature at the ends of conjugate diamete ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, then ,prove that

$$\rho_1^{\frac{2}{3}} + \rho_2^{\frac{2}{3}} = \frac{a^2 + b^2}{(ab)^{\frac{2}{3}}}$$

e) Determine the point of inflexion of curve

$$y^2 = x(x+1)^2 .$$

f) Find the evolute of the curve

$$x = a(\theta - \sin\theta),\ y = a(1 - \cos\theta).$$

GROUP - C

(Marks : 30)

Answer any *three* questions : 3 ×

12. a) Find the integrating factor and then solve :

$$y(xy + 2x^2y^2)\, dx + x(xy - x^2y^2)\, dy = 0 .$$

b) Solve by reducing to a linear equation :

$$(1 + x^2)\frac{dy}{dx} - 4x^2 \cos^2 y + x \sin 2y = 0 .$$

c) Find the orthogonal trajectories of the cardioides $r = a\,(1 - \cos\theta)$.

13. a) Transform the given equation to Clairaut's equation by putting $x^2 = u,\ y^2 = v$ and hence find the general and singular solution :

$$x^2(y - px) = p^2y, \text{ where } p = \frac{dy}{dx}.$$

b) Solve : $(x^2D^2 - 3xD + 5)y = x^2 \sin(\log x)$ where $D = \frac{d}{dx}$.

14. a) By the method of undetermined coefficient find the solution of the equation $(D^2 - 4D + 4)y = x^3 e^{2x} + x\, e^{2x}$. 5

b) Solve by the method of variation of parameters : $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + y = \dfrac{e^{-x}}{x^2}$. 5

15. a) Solve : $(1+x)^2 \dfrac{d^2y}{dx^2} + (1+x)\dfrac{dy}{dx} + y = 4\cos\,(\log\,(1+x))$. 5

b) Show that $\sin x \dfrac{d^2y}{dx^2} - \cos x \dfrac{dy}{dx} + 2y\sin\,x = 0$ is exact and solve it completely. 5

16. a) Solve $x\dfrac{d^2y}{dx^2} - (x+2)\dfrac{dy}{dx} + 2y = x^3 e^x$ after the determination of a solution of its reduced equation. 5

b) Reduce to normal form and hence solve :

$$x^2 \frac{d^2y}{dx^2} - 2(x^2 + x)\frac{dy}{dx} + (x^2 + 2x + 2)y = 0.$$ 5

17. a) Show that $x\dfrac{d^2y}{dx^2} + (x-1)\dfrac{dy}{dx} - y = x^2$, by the method of operational factors. 5

b) Solve : $(1+x)^2 \dfrac{d^2y}{dx^2} + (1+x)\cdot\dfrac{dy}{dx} + y = 4\cos\ \log\ (1+x)$ by changing the independent variable. 5

GROUP - D

(Marks : 25)

Answer any *five* questions. 5 × 5 = 25

18. If $\vec{a}, \vec{b}, \vec{c}$ be three unit vectors such that $\vec{a} \times \vec{b} \times \vec{c} = \dfrac{1}{2}\vec{b}$, find the angles which $\vec{a}$ makes with $\vec{b}$ and $\vec{c}$; $\vec{b}, \vec{c}$ being non-parallel. 5

19. a) A particle moves along the curve $x = t^3 + 1$, $y = t^2$, $z = 2t + 5$, where t den[otes] time. Find the component of velocity and acceleration at time $t = 1$ in [the] direction $\vec{i} + \vec{j} + 3\vec{k}$.

b) If $\frac{d}{dt}\vec{a} = \vec{c} \times \vec{a}$, $\frac{d}{dt}\vec{b} = \vec{c} \times \vec{b}$, then show [that] $\frac{d}{dt}\left(\vec{a} \times \vec{b}\right) = \vec{c} \times \left(\vec{a} \times \vec{b}\right)$. 3

20. a) Prove by vector method, the trigonometrical formula $\cos C = \frac{a^2 + b^2 - c^2}{2ab}$.

b) Show that $\left[\vec{\alpha} + \vec{\beta}, \vec{\beta} + \vec{\gamma}, \vec{\gamma} + \vec{\alpha}\right] = 2\left[\vec{\alpha}, \vec{\beta}, \vec{\gamma}\right]$. when [ ] denotes the sca[lar] triple product. 3

21. For any two vectors $\vec{a}$ and $\vec{b}$, prove that

$$\text{grad}\,(\vec{a} \cdot \vec{b}) = \vec{a} \times \text{curl}\,\vec{b} + \vec{b} \times \text{curl}\,\vec{a} + (\vec{a} \cdot \nabla)\,\vec{b} + (\vec{b} \cdot \nabla)\,\vec{a}$$

22. a) Find the constants a, b, c so that

$\vec{f} = (x + 2y + az)\,\vec{t} + (bx - 3y - z)\,\vec{j} + (4x + cy + 2z)\,\vec{k}$ is irrotational.

b) If the vector $\vec{f} = 3x\,\vec{i} + (x + y)\,\vec{j} - ax\,\vec{k}$ is solenoidal, find a. 3

23. a) A particle acted on by the constant forces $4\vec{i} + \vec{j} - 3\vec{k}$ and $3\vec{i} + \vec{j} - \vec{k}$, displaced from the point $\vec{i} + 2\vec{j} + 3\vec{k}$ to the point $5\vec{i} + 4\vec{j} + \vec{k}$. Find the wo[rk] done by the force on the particle.

b) Find the vector equation of the plane passing through the origin and parallel [to] the vectors $2\vec{i} + 3\vec{j} + 4\vec{k}$ and $4\vec{i} - 5\vec{j} + \vec{k}$. 3

24. a) If $\vec{\alpha}, \vec{\beta}, \vec{\gamma}$ be three vectors such that $\vec{\alpha} + \vec{\beta} + \vec{\gamma} = \vec{0}$ and $|\vec{\alpha}| = 2$, $|\vec{\beta}| = 4$, $|\vec{\gamma}| = 6$, then show that $\vec{\alpha} \cdot \vec{\beta} + \vec{\beta} \cdot \vec{\gamma} + \vec{\gamma} \cdot \vec{\alpha} = -28$.

b) Find the directional derivative of the function $\phi = x^2 - y^2 + 2z^2$ as $P(1, 2, 3)$ in the direction of the line PQ, where Q is $(5, 0, 4)$. 3 + 2

25. Show that the necessary and sufficient condition that a non-zero vector $\vec{u}$ always remains parallel to a fixed line is that $\vec{u} \times \frac{d\vec{u}}{dt} = \vec{0}$. 5

26. If $\vec{A} = 2xz^2 \vec{i} - yz \vec{j} + 3xz^3 \vec{k}$, $\phi = x^2 yz$, then find,

(i) curl $(\phi \vec{A})$

(ii) curl curl $\vec{A}$. 3 + 2